The Riemann Hypothesis

(for High School Graduates)

by

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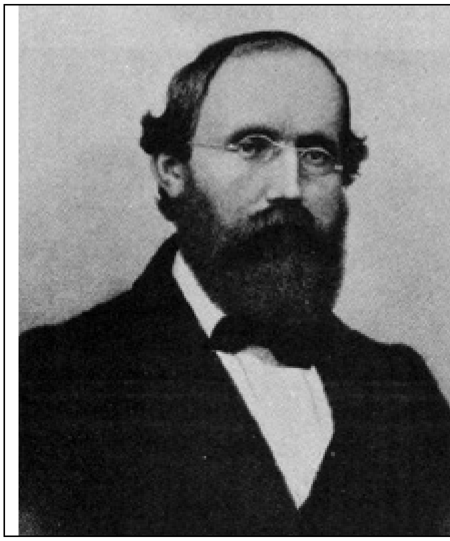
for the

Math and Science Lecture Series

at

College of the Desert



Bernhard Riemann (1826-1866)

- Studied under Gauss and Weber at Göttingen
- Friends with Dedekind and Dirichlet
- Uncanny knack for visualizing space
- Laid foundation for Relativity theory
- Refined definition for integral
- Studied the zeta function

- Very shy, only at ease with his family and a few mathematicians
- Very pious, in the Lutheran sense
- Very philosophical, with a vivid geometrical imagination
- Hypochondriac

From http://mathworld.wolfram.com/UnsolvedProblems.html

Unsolved Problems

There are many unsolved problems in mathematics. Some prominent outstanding unsolved problems (as well as some which are not necessarily so well known) include

1. The Goldbach conjecture.

2. The Riemann hypothesis.

3. The conjecture that there exists a Hadamard matrix for every positive multiple of 4.

4. The twin prime conjecture (i.e., the conjecture that there are an infinite number of twin primes).

5. Determination of whether NP-problems are actually P-problems.

6. The Collatz problem.

7. Proof that the 196-algorithm does not terminate when applied to the number 196.

8. Proof that 10 is a solitary number.

9. Finding a formula for the probability that two elements chosen at random generate the symmetric group S_n.

10. Solving the happy end problem for arbitrary n.

Riemann Hypothesis

Version 1:

The non-trivial complex zeros of the zeta function $\zeta(z)$ lie on the line $\mathrm{Re}(z) = \frac{1}{2}$.

Version 2:

Begin with the set of all natural numbers $\{1, 2, 3\ldots\}$, discard all those that are divisible by the square of any integer greater than 1.

Thus throw out 4, 8, 9, 16, 18, 20, 24,…, etc.

We're left with the list of *squarefree* positive integers,

1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, … .

The factorization of any one of these will contain no prime twice: 2*3*5*7 = 210 would be on the list, for example.

Squarefree numbers are either the product of an even or an odd number of prime factors.

Let's say squarefree numbers with an odd number of prime factors are blue, the rest are red. Thus 14 is red and 30 is blue. 18 is colorless because it's not squarefree.

The squarefree numbers ≤ 71 are

1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 35, 37, 38, 39, 41, 42, 43, 46, 47, 51, 53, 55, 57, 58, 59, 61, 62, 65, 66, 67, 69, 70, 71

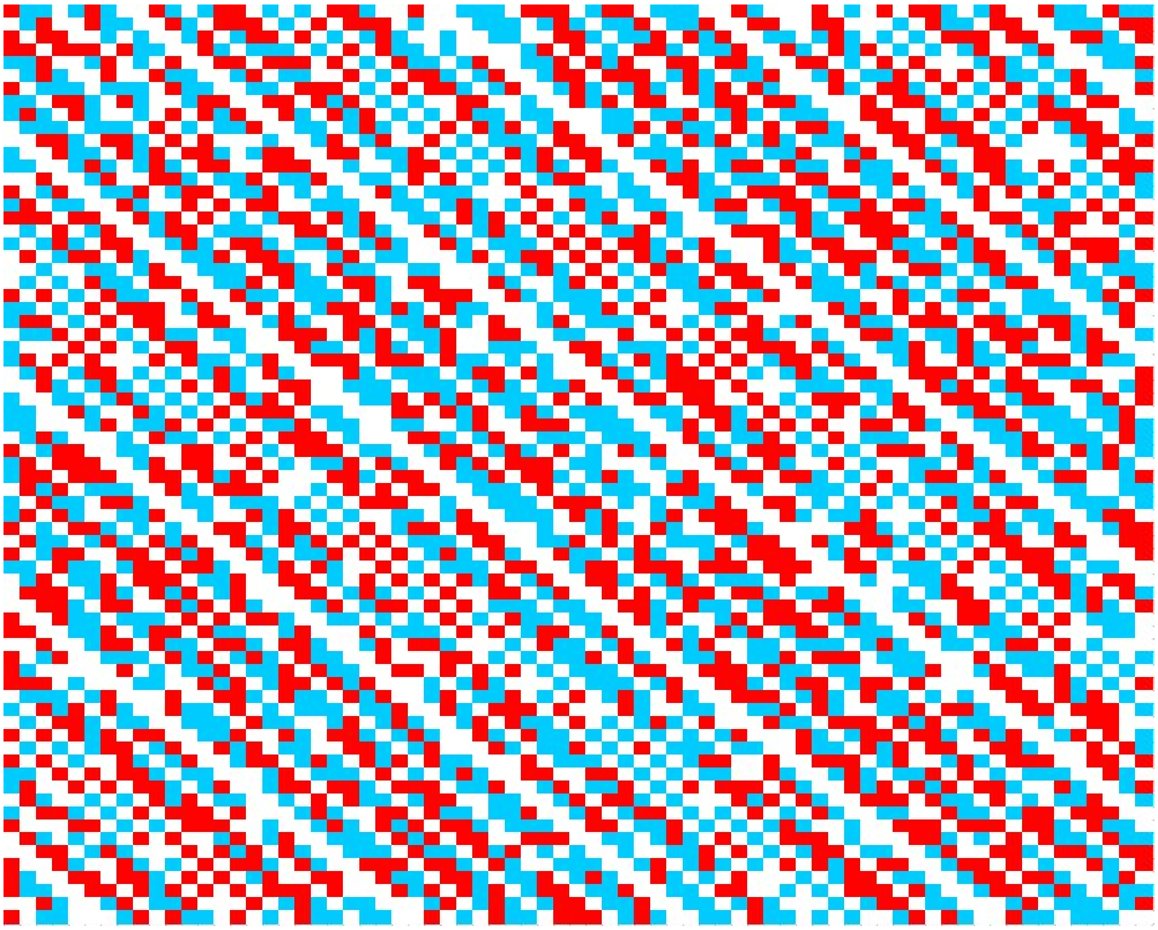
Of these, there are 24 blue numbers,
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 30, 31, 37, 41, 42, 43, 47, 53, 59, 61, 66, 67, 70, 71
and 20 red numbers:
1, 6, 10, 14, 15, 21, 22, 26, 33, 35, 28, 29, 26, 51, 55, 57, 58, 62, 65, 69

Thus, among the first 71 positive integers, there are 4 more blue numbers than red. The Riemann's hypothesis says roughly that in every interval [1, *n*] there are not very different quantities of red and blue numbers. More precisely, not in Riemann's formulation, but in a fully equivalent form more approachable by a high school student:

RIEMANN'S HYPTOTHESIS: *Fix* ε>0. Then we can find N such that for all $n > N$ the number of blue numbers in $[1, n]$ does not differ from the number of red numbers in $[1, n]$ by more than $n^{1/2+\varepsilon}$.

That is, the disparity between red and blue is at most 'about' $\sqrt{n}$.
For instance $4 < \sqrt{71} \approx 8.4$

Below is a 71 by 71 grid showing the colors of number 1,2,…,71 in the first row, 72, 73,…,142 in the second and on to 4970,4971,…,5041 in the last. There are 1547 blues and 1535 reds. The difference of 12 is much less than 71.



```
=IF(AND(Sheet3!CB7=1,MOD(NumPrimeFactors(Sheet3!G7),2)=0),1,
    IF(AND(Sheet3!CB7=1,MOD(NumPrimeFactors(Sheet3!G7),2)=1),2,3))
```

The Zeta Function

If $\operatorname{Re}(s) > 1$ then $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} + \cdots$

The **Harmonic Series**, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is a special case of the zeta function, $\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$ is easy to prove using just ordinary arithmetic. One of the earliest proofs was by French scholar Nicole d'Oresme (1323-1382) who noted that

$$\frac{1}{3} + \frac{1}{4} > 2\left(\frac{1}{4}\right) = \frac{1}{2}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 4\left(\frac{1}{8}\right) = \frac{1}{2}$$

$$\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} > 8\left(\frac{1}{16}\right) = \frac{1}{2}$$

Through analytic continuation, the Zeta function's domain can be extended to all complex numbers except $z = 1$:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s)$$

The Basel Problem

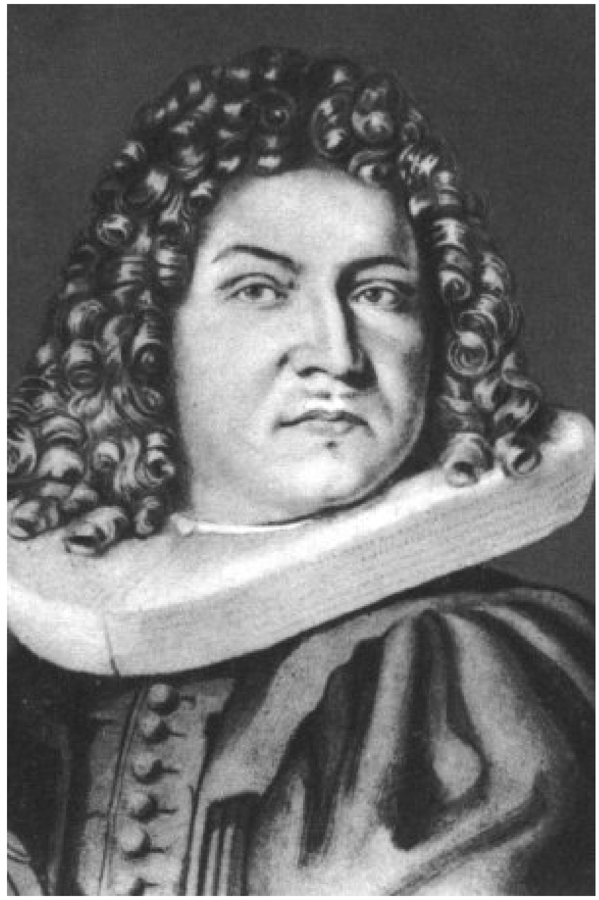
First stated by Jacob Bernoulli (1654–1705) in 1689:

Find a closed form for

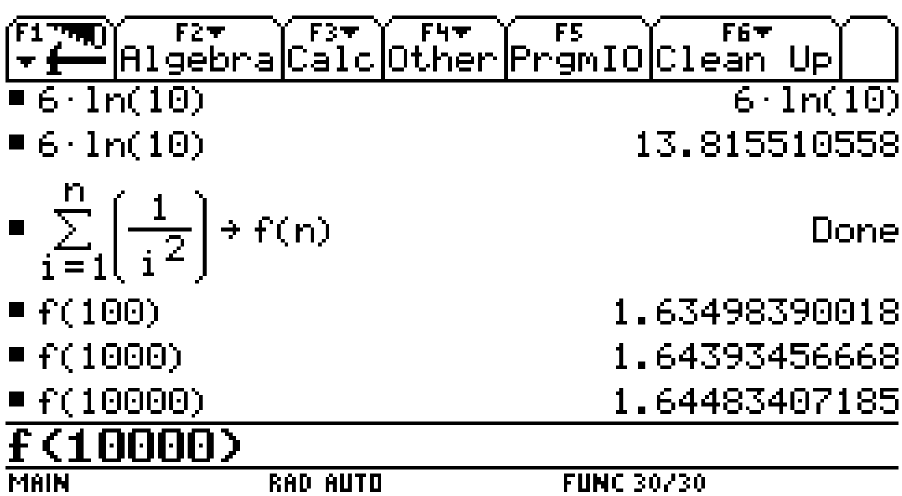
$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots$$

Note: "closed form" is an imprecise phrase meaning, loosely, "able to be expressed without using a limit, infinity or the dot, dot, dot…"

Sometimes, simply awarding a special notation, like $\sqrt{2}$ to represent the open form is sufficient.



Here's a screen capture of my work approximating $\zeta(2)$ on the TI-Voyage 200. I create a function for the n^{th} partial sum $f(n) = \sum_{i=1}^{n} \frac{1}{i^2}$ and then evaluate $f(100), f(1000)$ and $f(10000)$. The calculator took about 4 hours or so to cough these up.



The last approximation, 1.64483 is still 0.006% short of the convergent value, which is closer to 1.64493406685…, which is still an "open form" since it is not an exact representation and requires the dot, dot, dot – the ellipsis.

The Basel Problem was solved by Leonhard Euler in 1735, who astonished the world with $\zeta(2) = \frac{\pi^2}{6}$. In fact, based on this result, we can compute $\zeta(N)$ for all even values of *N*. For instance,

$$\zeta(4) = \frac{\pi^4}{90}, \ \zeta(6) = \frac{\pi^6}{945}$$

If *N* is odd then $\zeta(N)$ is still mysterious. It wasn't until 1978 that Apéry's number $\zeta(3) \approx 1.202$ was proved irrational, by none other than the eponymous Apéry! The ashes of Roger Apéry are stored with those of his parents in columbarium number 7972 at the *Père Lachaise* cemetery in Paris (France) behind a plaque where his most famous result is engraved this way:

$$1 + 1/8 + 1/27 + 1/64 + \ldots \neq p/q$$


LOUISE VAN DER CRUYSSEN
ÉPOUSE GEORGES APÉRY
1892 – 1965
GEORGES APÉRY
1887 – 1978
ROGER APÉRY
1916 – 1994
$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \dots \neq \frac{p}{q}$

Traditional Fourfold Division of Mathematics into Sub-disciplines:

- *Arithmetic*—The study of whole numbers and fractions.
  Typical theorem: The product of two odd numbers is odd.

- *Geometry*—They study of figures in space—points, lines, curves, and three-dimensional objects. Typical theorem: The base angles of an isosceles triangle are congruent.

- *Algebra*—The use of abstract symbols to represent mathematics objects (numbers, lines, matrices, transformations), and the study of the rules for combining those symbols.
  Sample theorem: We can factor a difference of squares:
  $x^2 - y^2 = (x+y)(x-y)$.

- *Analysis*—The study of limits. Sample theorem: The harmonic series is divergent.

Riemann helped bring about the "great fusion" of 19th century: The cross-breeding of arithmetic and analysis to create analytic number theory. This a dichotomy of measurement built right into the English language: How much? How many? Can we measure the same sorts of things we count on a continuum? The natural numbers are embedded in the real numbers, but, like the rationals, they're islands set apart from one another in a way that irrationals are not? Note to self: find out what a Dedekind cut is.

The Prime Number Theorem

How many primes are then less than a given number?

Definition: A prime number is a natural number greater than 1 that is divisible only by 1 and itself.

The first 100 primes can be found using the simple Mathematica command

```
For[i=1,i<100,Print[Prime[i]];i++]
```

2	3	5	7	11	13	17	19	23	29	31	37
41	43	47	53	59	61	67	71	73	79	83	89
97	101	103	107	109	113	127	131	137	139	149	151
157	163	167	173	179	181	191	193	197	199	211	223
227	229	233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349	353	359
367	373	379	383	389	397	401	409	419	421	431	433
439	443	449	457	461	463	467	479	487	491	499	503
509	521	523									

Let $\pi(x)$ be the number of primes less than x. Then we can start tabulating:

x	25	50	75	100	125	150	175	200	225	250	275	300	325	350	375	400	425	450	475	500	525
$\pi(x)$	9	16	21	25	30	35	40	46	48	53	58	62	66	70	74	78	82	86	91	95	99

It's easy to verify there are 168 primes less than 1000, so $\pi(1000) = 168$.

The rate of occurrence of primes seems to decrease.

In fact, while 16.8% of natural numbers less than 1000 are prime, we can use Mathematica to compute the 4 primes leading up to and including the billionth prime with the command

```
{Prime[10^9-4],Prime[10^9-3],Prime[10^9-2],Prime[10^9-1],Prime[10^9]}
```

And here they are:

```
{22801763389, 22801763459, 22801763471, 22801763477, 22801763489}
```

Note that the gaps between primes are larger and that the proportions has shrunk to about 4.4%

Do the primes thin out to nothing?

No, Euclid (~314 BCE) showed $1 \times 2 \times 3 \times \cdots \times N + 1$ is not divisible by any number from 1 to *N*, so it's smallest prime factor must be larger than *N*.

Can we find a rule that describes how the density of primes gets smaller?

N	How many primes less than N? $\pi(N)$
1,000	168
1,000,000	78,498
1,000,000,000	50,847,534
1,000,000,000,000	37,607,912,018
1,000,000,000,000,000	29,844,570,422,669
1,000,000,000,000,000,000	24,739,954,287,740,860

Experimenting with different expressions involving *N* and $\pi(N)$ you might arrive at this:

N	$N/\pi(N)$
1,000	5.9524
1,000,000	12.7392
1,000,000,000	19.6666
1,000,000,000,000	26.5901
1,000,000,000,000,000	33.5069
1,000,000,000,000,000,000	40.4204

Note the relatively steady (nearly linear) increase!!!!

A Quick Review of Things Exponential and Logarithmic

$e \approx 2.718281828459045235360287\ldots$

N	3^N
1	3
2	9
3	27
4	81

N	e^N
1	2.71828182
2	7.38905609
3	20.08553692
4	54.59815003

N	$\log(N)$
2.718281828	1
7.389056099	2
20.08553692	3
54.59815003	4

N	$\log(N)$	$N/\pi(N)$	% error
1,000	6.9078	5.9524	16.05%
1,000,000	13.8155	12.7392	8.45%
1,000,000,000	20.7233	19.6666	5.37%
1,000,000,000,000	27.6310	26.5901	3.91%
1,000,000,000,000,000	34.5388	33.5069	3.08%
1,000,000,000,000,000,000	41.4465	40.4204	2.54%

<u>The</u> Prime Number Theorem (PNT)

$$\pi(N) \sim \frac{N}{\log(N)}$$

This means that the probability that an arbitrarily chosen natural number is prime is

$$\frac{\pi(N)}{N} \sim \frac{1}{\log(N)}$$

and that the N^{th} prime number is $\sim N\log(N)$. These are just ball park figures. For instance, the millionth prime number is 15,485,863 while $10^6 \log\left(10^6\right) \approx 13,815,511$. The error in approximation is almost 11%

Review of power rules:

Power Rule 1: $x^m \times x^n = x^{m+n}$

Power Rule 2: $x^m \div x^n = x^{m-n}$

Power Rule 3: $\left(x^m\right)^n = x^{m\times n}$

Power Rule 4: $x^0 = 1$, for any positive x

Power Rule 5: $x^{-n} = \dfrac{1}{x^n}$

Power Rule 6: $x^{\frac{m}{n}}$ is the n^{th} root of x^m.

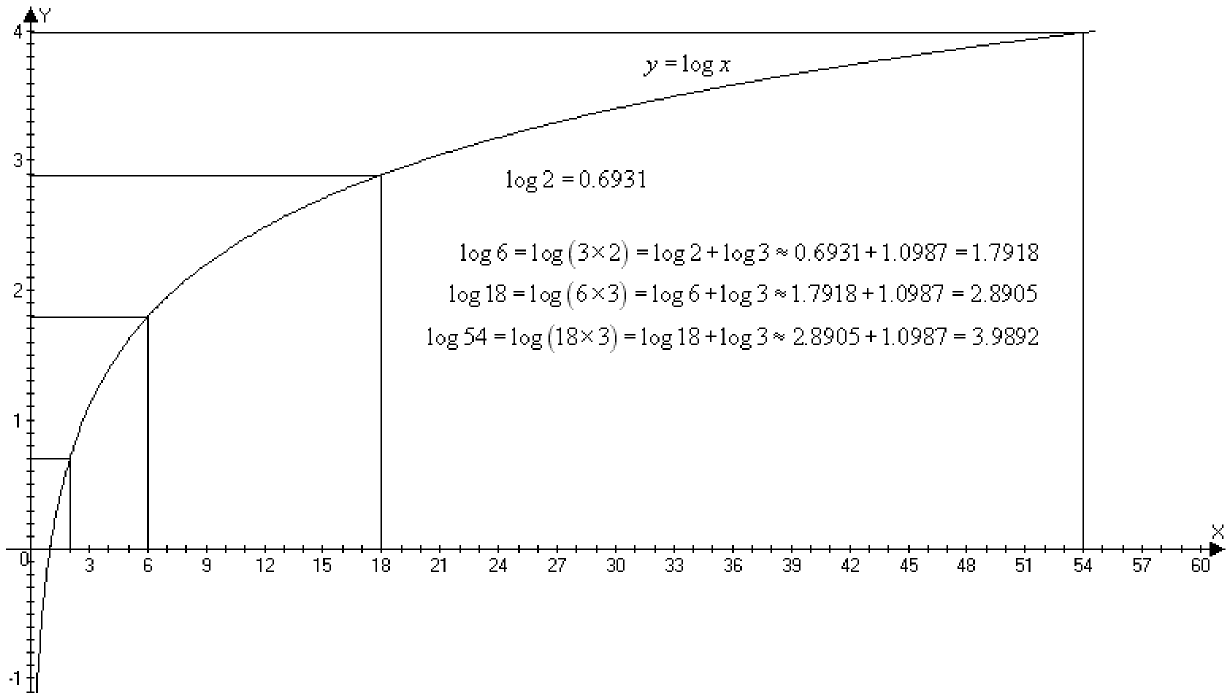
Power Rule 7: $\left(x\times y\right)^n = x^n \times y^n$

Power Rule 8: $x = e^{\log x}$

$a\times b = e^{\log a}\times e^{\log b} = e^{\log a+\log b} = e^{\log(a\times b)}$

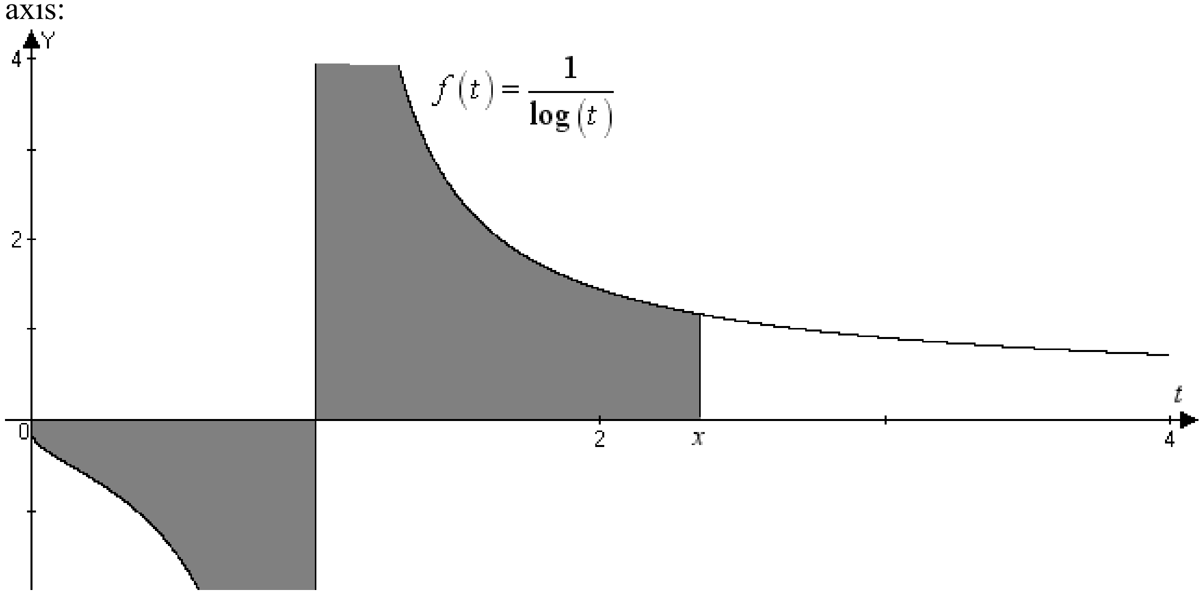
Power Rule 9: $\log\left(a\times b\right) = \log a + \log b$

Power Rule 10: $\log\left(a^N\right) = N\times\log\left(a\right)$



The diagram above illustrates how logarithms convert harder multiplication computations to easier addition computations; repeated multiplication by 3 becomes repeated addition of log3.

Consider the area between reciprocal of the log function and the interval [0, x] on the axis:



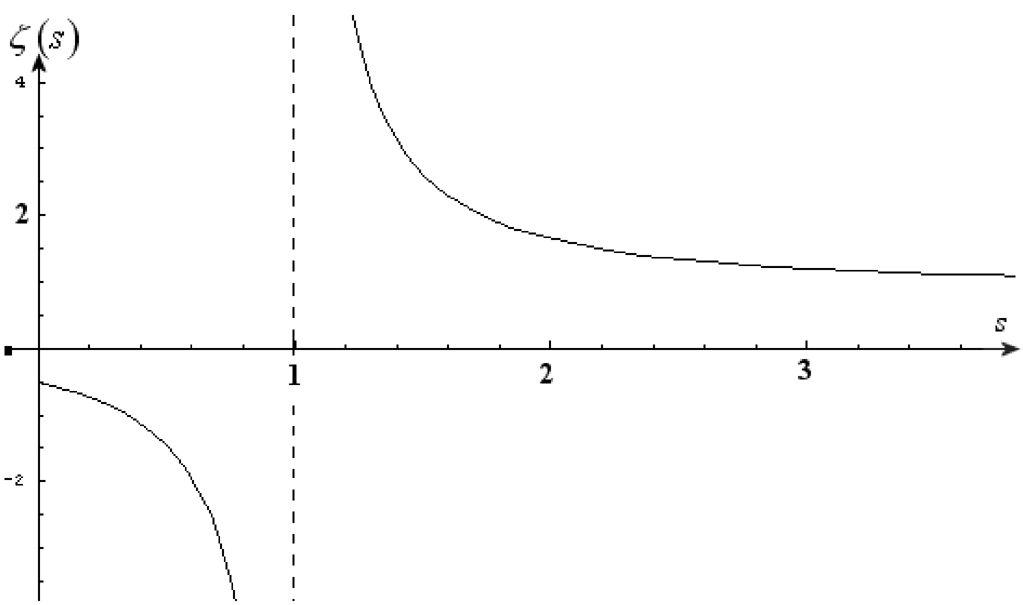
The log integral function is $Li(x) = \int_0^x \frac{1}{\log(t)} dt$ and gives the shaded area…which depends on x. I think it turns out this integral is the same as $Li(x) = \int_2^x \frac{1}{\log(t)} dt$ so you can skip the singularity.

It turns out that $Li(x) \sim \frac{N}{\log N}$ so $\pi(N) \sim Li(N)$

Back to the Zeta Function

How does $\zeta(s)$ depend on s?

- $\zeta(1)$ is undefined (infinite.)
- We have nifty closed form formulas for $\zeta(2)$, $\zeta(4)$, $\zeta(6)$,… but not other s values.
- $\zeta(1.0001) \approx 10{,}000.577222...$ In fact, Zeta approaches a vertical asymptote.
- Mathematica command:
  ```
  Plot[Zeta[x],{x,0,4},PlotRange->{-5, 5}]
  ```
  produces:



The Geometric Series

If $S_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$ then

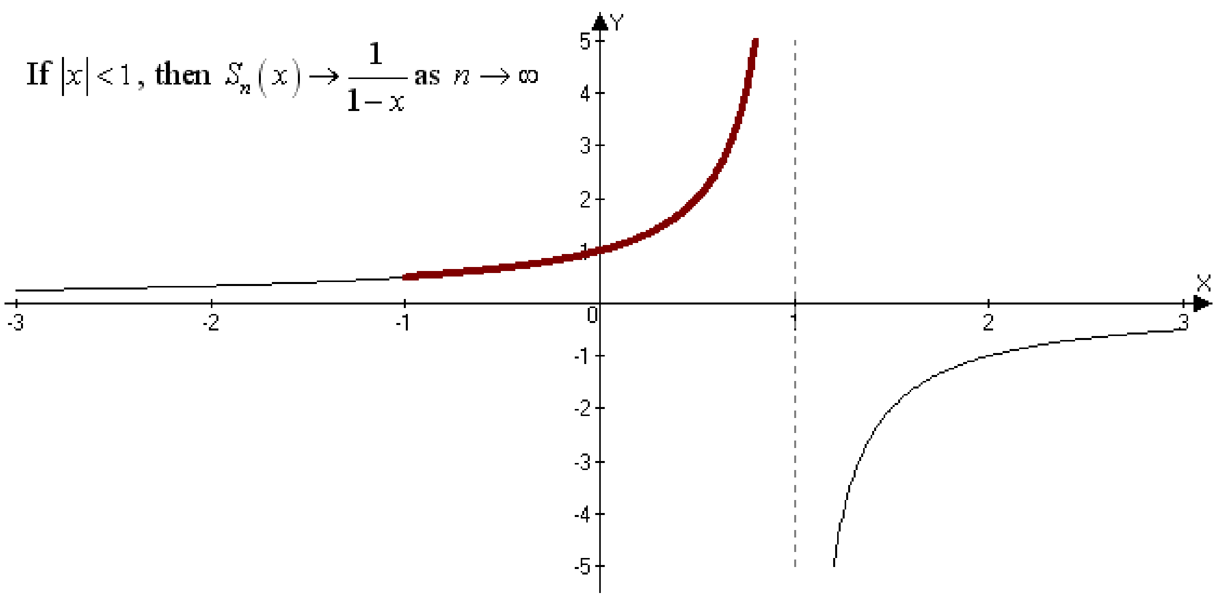
$$\begin{aligned} S_n(x) - xS_n(x) &= 1 + x + x^2 + \cdots + x^n \\ &\quad - \left(x + x^2 + \cdots + x^n + x^{n+1}\right) \end{aligned}$$

$$\Leftrightarrow (1-x)S_n(x) = 1 - x^{n+1}$$

$$\Leftrightarrow S_n(x) = \frac{1-x^{n+1}}{1-x}$$

If $|x| < 1$ the $x^{n+1} \to 0$ as $n \to \infty$ so that

$$\text{If } |x| < 1, \text{ then } S_n(x) \to \frac{1}{1-x} \text{ as } n \to \infty$$



The Golden Key

Recall the zeta function for $s > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} + \cdots$$

Multiply both sides by $\frac{1}{2^s}$ (power rule 7):

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \frac{1}{12^s} + \frac{1}{14^s} + \frac{1}{16^s} + \frac{1}{18^s} + \cdots$$

Now subtract the second expression from the first:

$$\zeta(s) - \frac{1}{2^s}\zeta(s) = \left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{15^s} + \frac{1}{17^s} + \frac{1}{19^s} + \cdots$$

Do it again for multiples of 3. Multiply both sides by $\frac{1}{3^s}$ to get

$$\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \frac{1}{33^s} + \frac{1}{39^s} + \frac{1}{45^s} + \cdots$$

and subtract from the last difference to get

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) - \frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) =$$
$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \frac{1}{19^s} + \frac{1}{23^s} + \frac{1}{25^s} + \frac{1}{29^s} + \cdots$$

One more time:

$$\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) =$$
$$1 + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \frac{1}{19^s} + \frac{1}{21^s} + \frac{1}{23^s} + \frac{1}{29^s} + \cdots$$

Continuing this process ad infinitum:

$$\cdots\left(1-\frac{1}{17^s}\right)\left(1-\frac{1}{13^s}\right)\left(1-\frac{1}{11^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s)=1$$

…and solving for zeta:

$$\zeta(s)=\frac{1}{1-\frac{1}{2^s}}\times\frac{1}{1-\frac{1}{3^s}}\times\frac{1}{1-\frac{1}{5^s}}\times\frac{1}{1-\frac{1}{7^s}}\times\frac{1}{1-\frac{1}{11^s}}\times\frac{1}{1-\frac{1}{13^s}}\times\frac{1}{1-\frac{1}{17^s}}\times\frac{1}{1-\frac{1}{19^s}}\times\cdots$$

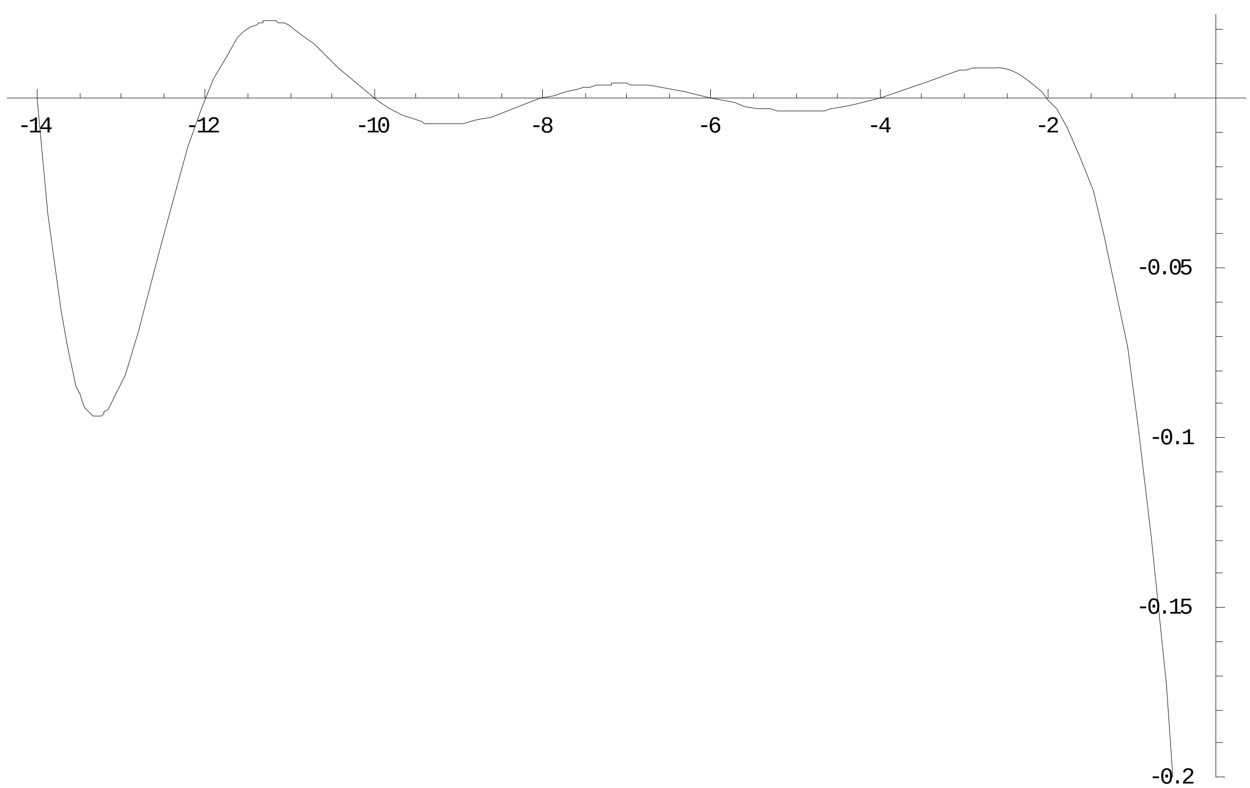
or

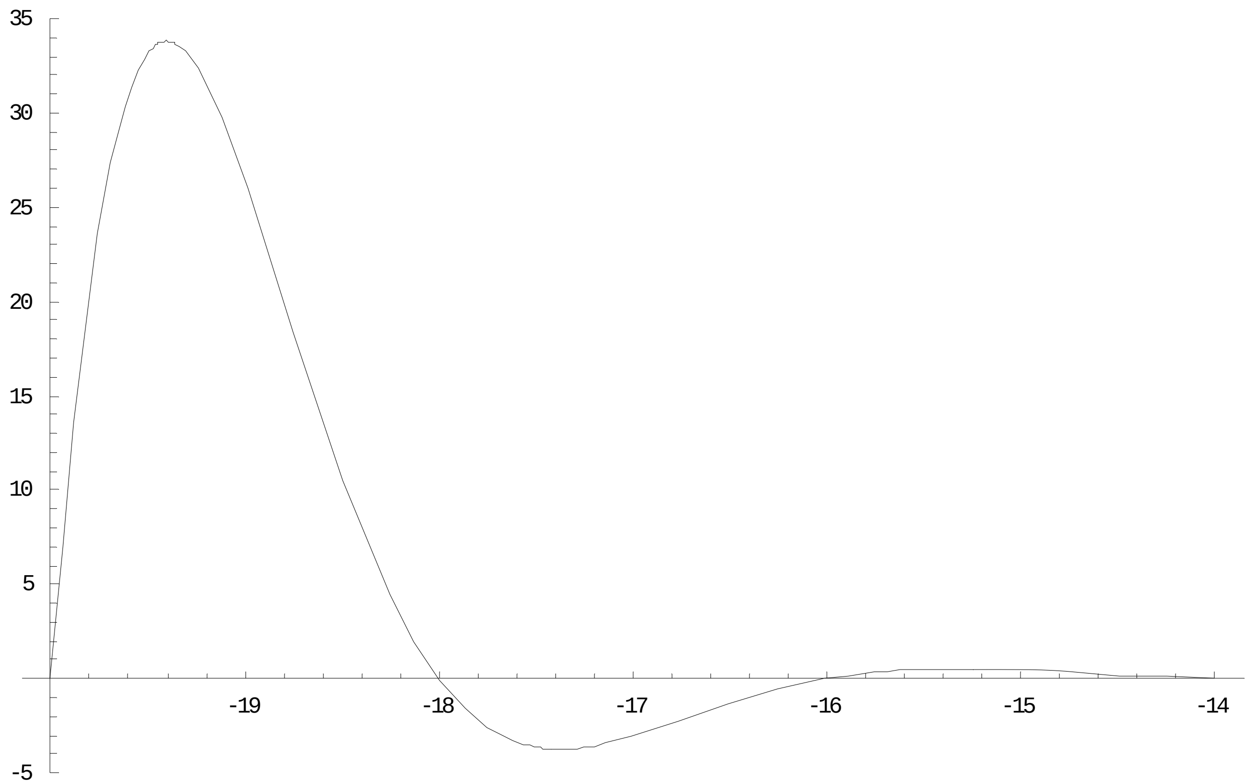
$$\zeta(s)=\left(1-2^{-s}\right)^{-1}\left(1-3^{-s}\right)^{-1}\left(1-5^{-s}\right)^{-1}\left(1-7^{-s}\right)^{-1}\left(1-11^{-s}\right)^{-1}\left(1-13^{-s}\right)^{-1}\cdots$$

or

$$\zeta(s)=\prod_{\text{primes}}\left(1-p^{-s}\right)^{-1}$$

The analytic continuation of the zeta function to values less than $s = 1$ is analogous to the continuation of the geometric series.




35
30
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-17
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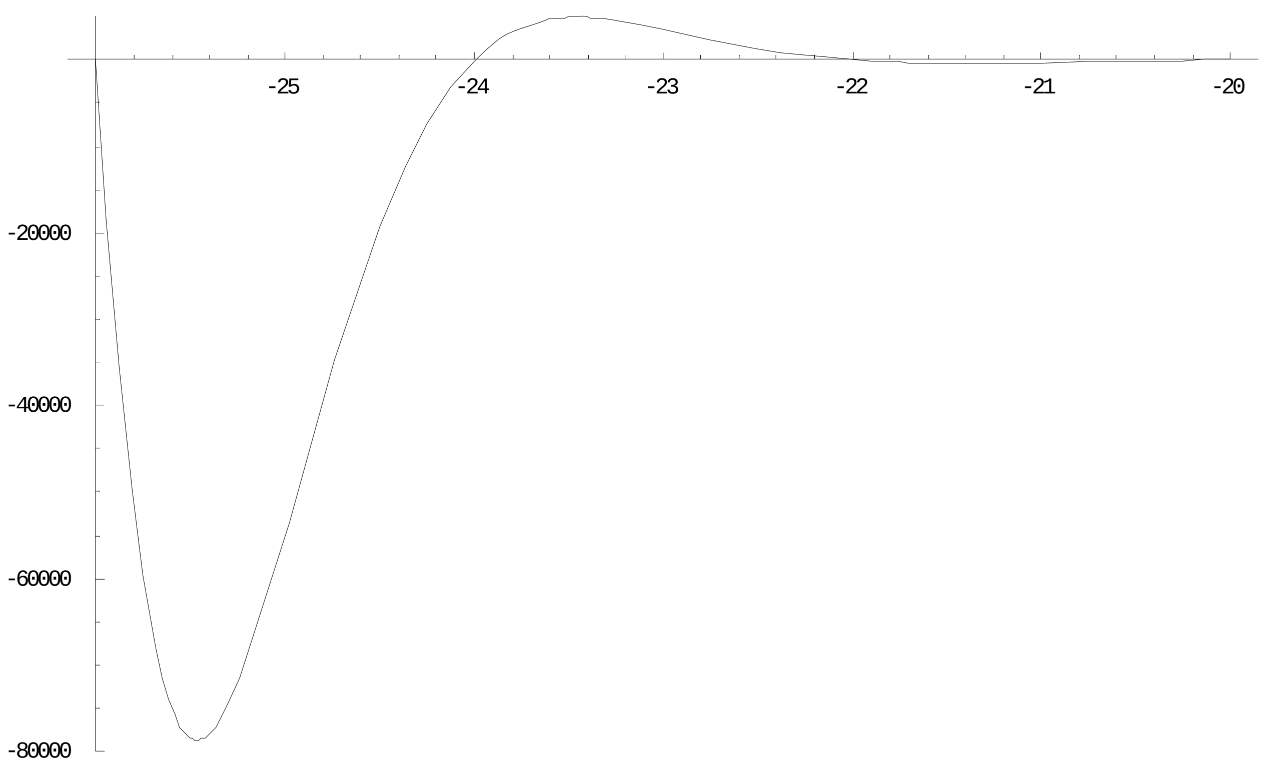
-15
-14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-24
-23
-22
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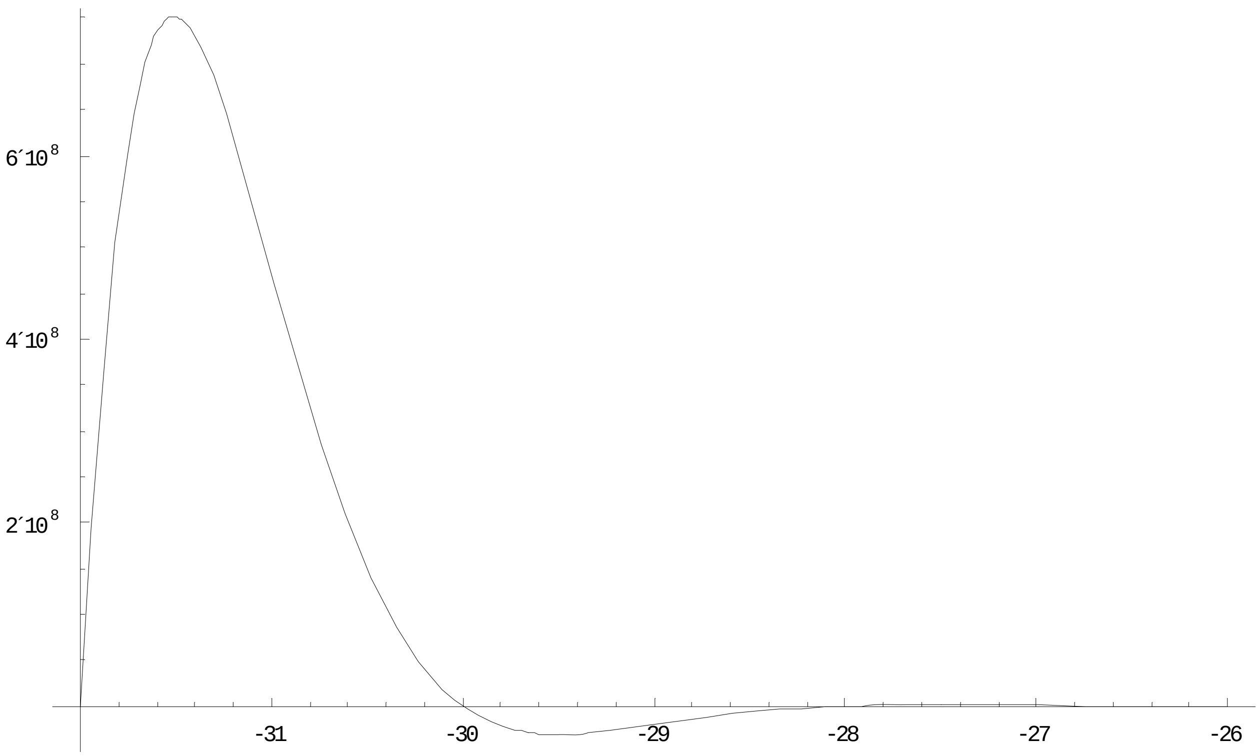
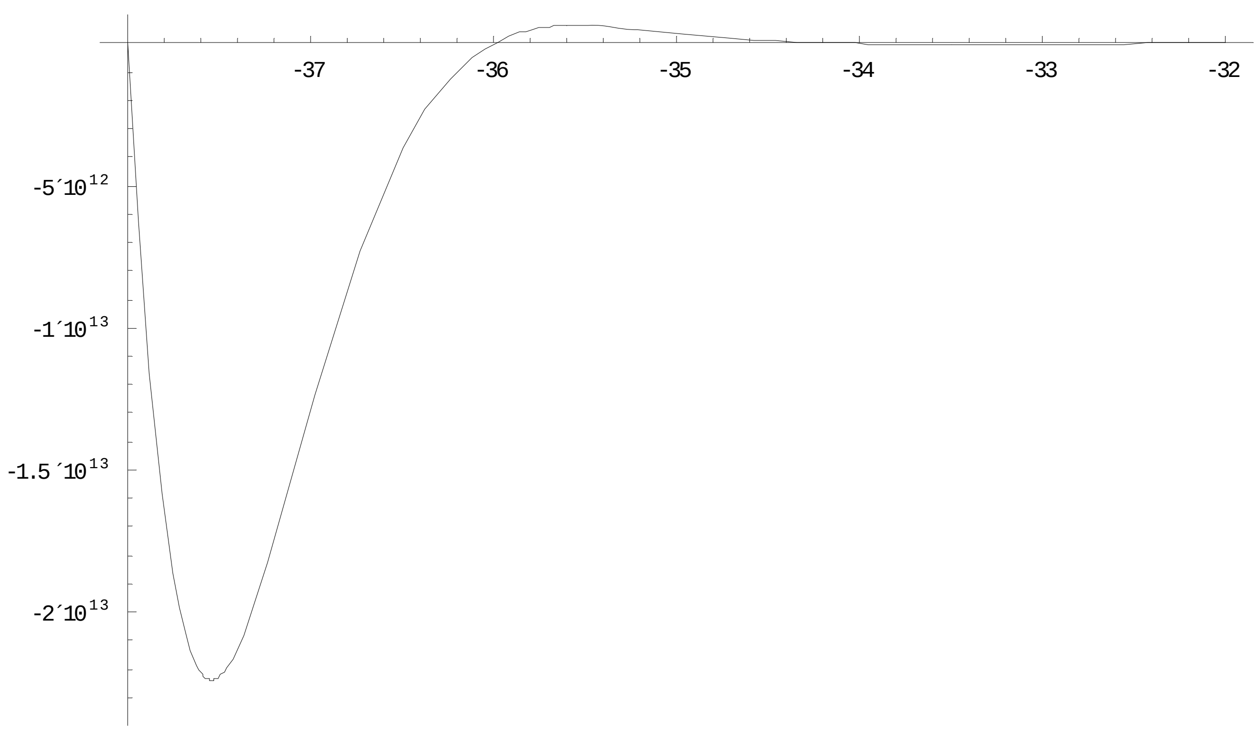
-20
-20000
-40000
-60000
-80000



6´10^8
4´10^8
2´10^8
-31
-30
-29
-28
-27
-26



-37
-36
-35
-34
-33
-32
-5´10^12
-1´10^13
-1.5´10^13
-2´10^13

$\eta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \frac{1}{7^s} - \frac{1}{8^s} + \frac{1}{9^s} - \cdots$ is convergent for $0 < s < 1$.

Now,

$$\begin{aligned}
\eta(s) &= \left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} + \cdots\right) \\
&\quad - 2 \times \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \frac{1}{12^s} + \cdots\right) \\
&= \left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} + \cdots\right) \\
&\quad - 2 \times \frac{1}{2^s}\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots\right) \\
&= \left(1 - 2 \times \frac{1}{2^s}\right)\zeta(s)
\end{aligned}$$

Solving for $\zeta(s)$:

$$\zeta(s) = \eta(s) \div \left(1 - \frac{1}{2^{s-1}}\right)$$

This allows us to compute values for $\zeta(s)$ between 0 and 1.

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)$$

where $\Gamma(s) = (s-1)!$, or an analytic extension of the factorial.

This allows you to compute, say,

$\zeta(-3) = \zeta(1-4) = 2^{-3}\pi^{-4}\cos(2\pi)\Gamma(4)\zeta(4) = \frac{1}{8} \times \frac{1}{\pi^4} 1 \times 3! \times \frac{\pi^4}{90} = \frac{1}{120} = 0.008\overline{3}$

Also, $\zeta(s)$ = 0 for all negative, even *s*.

What about unreal values of *s*?

Definition: The *imaginary unit*, ***i***, is the number whose square is –1: $i^2 \equiv -1$.

The set of *complex numbers* is $\mathbb{C} \equiv \{a + bi \mid a, b \in \mathbb{R}\}$ so that

$$\mathbb{C} \supset \mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z} \supset \mathbb{N}$$

Recall the geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$$

If $x = \frac{1}{2}i$ then

$$\frac{1}{1-\frac{1}{2}i} = 1 + \frac{1}{2}i + \frac{1}{4}i^2 + \frac{1}{8}i^3 + \frac{1}{16}i^4 + \frac{1}{32}i^5 + \frac{1}{64}i^6 + \cdots$$

$$= 1 + \frac{1}{2}i - \frac{1}{4} - \frac{1}{8}i + \frac{1}{16} + \frac{1}{32}i - \frac{1}{64} - \cdots$$

$$= 0.8 + 0.4i$$

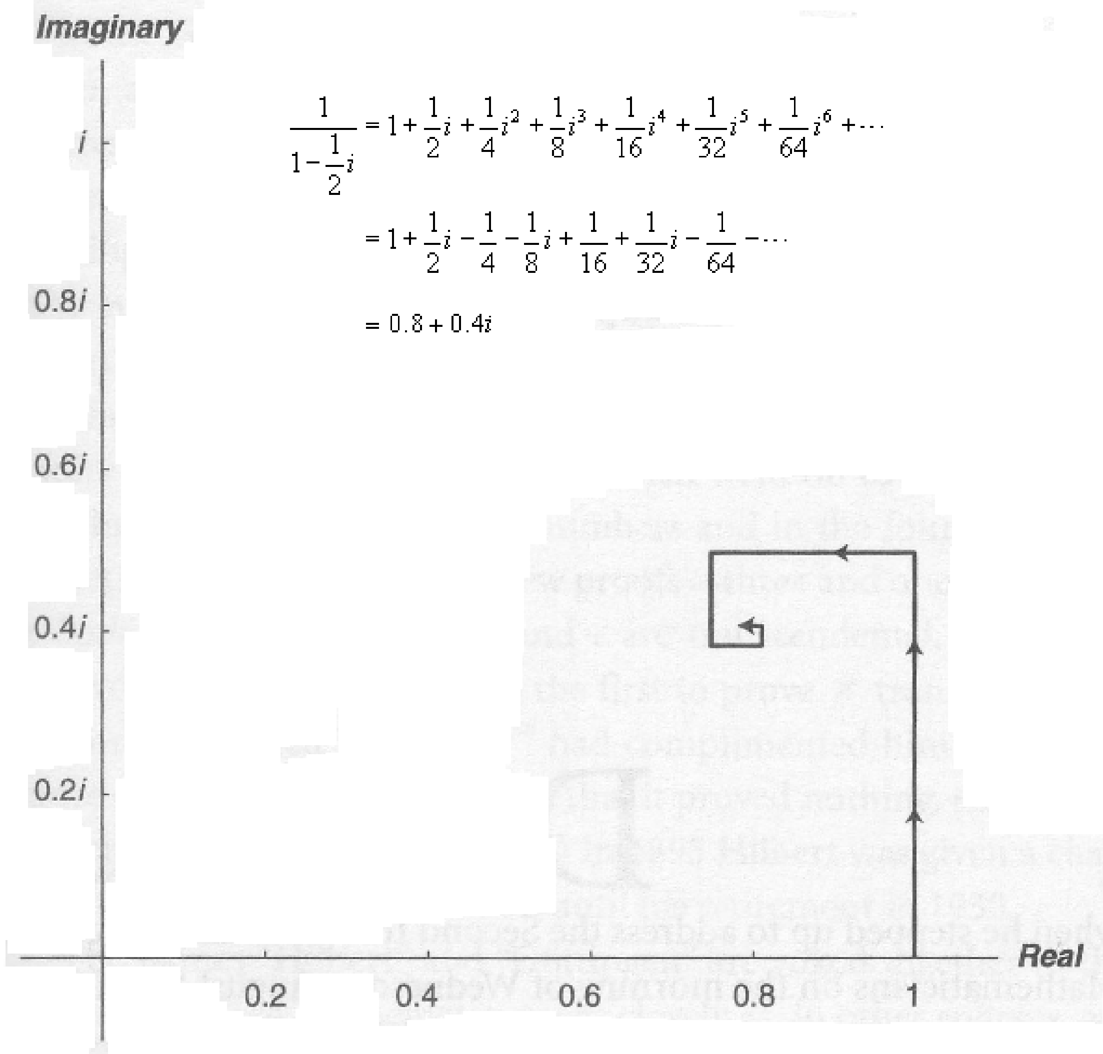


FIGURE 11-3 Analysis in the complex plane.

Now it turns out that

$$-\log(1-x)=\int\frac{1}{1-x}dx=\int\left(1+x+x^2+x^3+\cdots\right)dx=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$$

So that, for instance,

$$\log(2)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$$

Now, as everyone knows,

$$e^{i\pi}=-1$$

How is it possible to define a complex power of *e*, or any other number? By series, of course:

$$e^s=1+s+\frac{s^2}{1\times 2}+\frac{s^3}{1\times 2\times 3}+\frac{s^4}{1\times 2\times 3\times 4}+\frac{s^5}{1\times 2\times 3\times 4\times 5}+\cdots$$

$$e^{\pi i}\approx 1+3.141592i-\frac{9.869604}{2}-\frac{31.006277i}{6}+\frac{97.409091}{24}+\frac{306.001985i}{120}+\cdots$$

Since

$$\text{If } e^s=w \text{ then } s=\log w$$

We can take logarithms of complex numbers too.

…and since

$$a^z=\left(e^{\log a}\right)^z=e^{z\log a}$$

we can raise any complex number to a complex power.

To raise $-4+7i$ to the power $2-3i$, start by computing

$$\log a=\log(-4+7i)\approx 2.08719+2.08994i\,,$$

then multiply that by $2-3i$ to get

$$z\log a\approx(2-3i)(2.08719+2.08994i)\approx 10.442-2.08169i$$

and then raise *e* to that power:

$$(-4+7i)^{2-3i}=e^{z\log a}\approx e^{10.442-2.08169i}\approx -16793.46-29959.40i$$

Thus we can extend the domain of $\zeta(s)$ to the complex numbers…. $s\neq 1$

But it's hard to visualize a graph in 4-space.

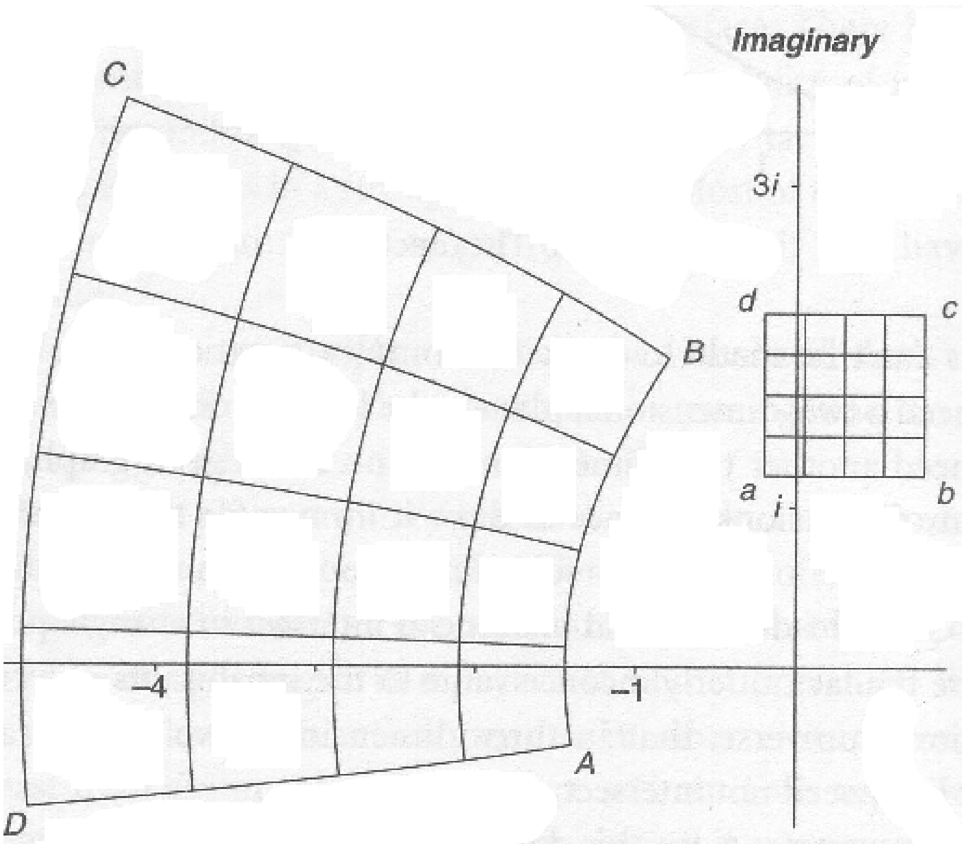
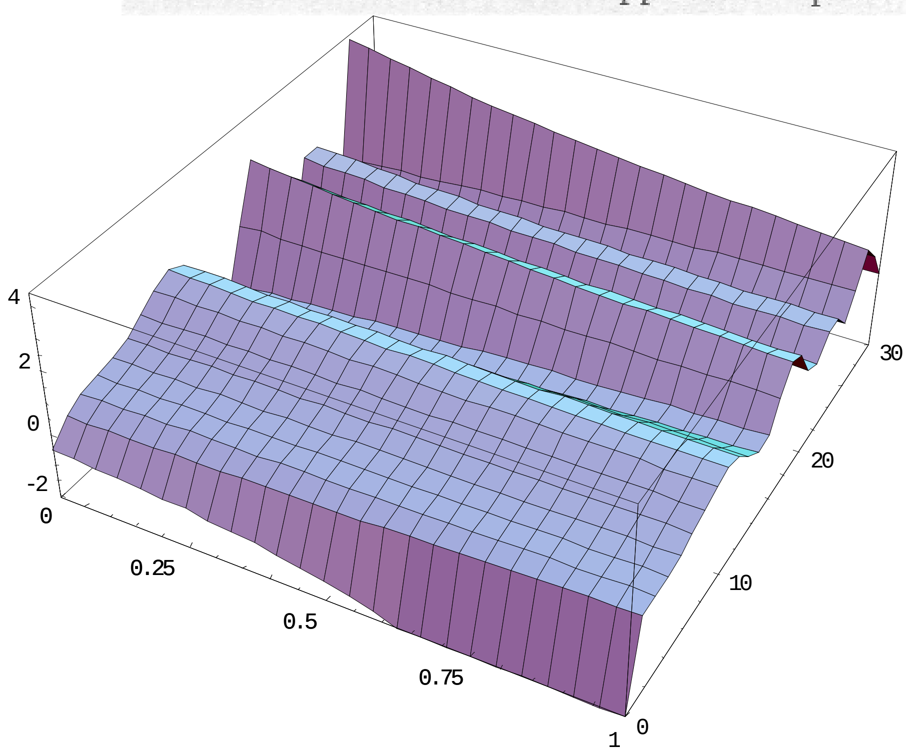
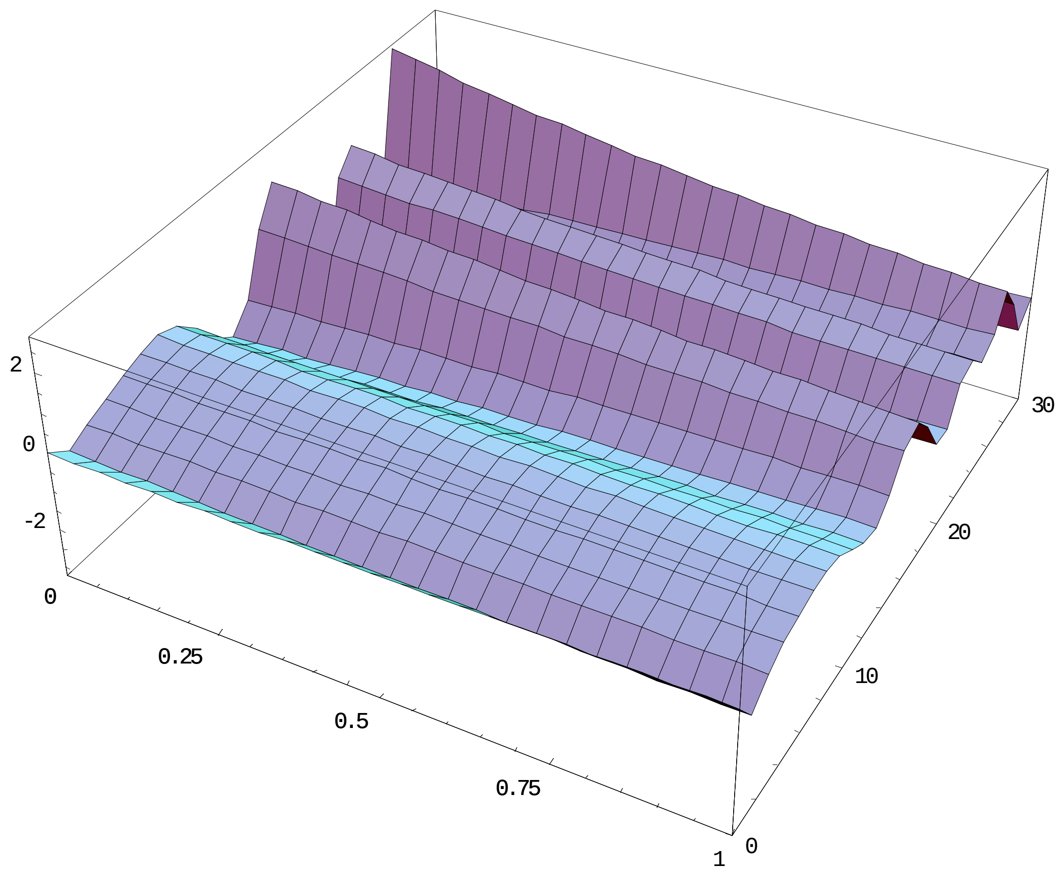


FIGURE 13-2 The function z^2 applied to a square.




```
Plot3D[Re[Zeta[x+I*y]],{x,0,1},{y,0,30}]
```




```
Plot3D[Im[Zeta[x+I*y]],{x,0,1},{y,0,30}]
```

Back to the Golden Key

Turn the Golden Key upside down:

$$\frac{1}{\zeta(s)}=\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{11^s}\right)\left(1-\frac{1}{13^s}\right)\cdots$$

Multiplying this out leads to infinitely many terms in a sum.

Each term is the product of either a 1 or a $\frac{1}{p^s}$ plucked from each factor.

First term: pluck 1 from each parenthesis: $1\times1\times1\times1\times1\times1\times1\times1\times1\times1\times1\cdots=1$

Second term: pluck 1 from every parenthesis except the first:

$$-\frac{1}{2^s}\times1\times1\times1\times1\times1\times\cdots=-\frac{1}{2^s}$$

Third term: pluck 1 from every parenthesis except the second:

$$-\frac{1}{3^s}\times 1\times 1\times 1\times 1\times 1\times\cdots=-\frac{1}{3^s}$$

Proceeding this way, this first infinite number of terms in the expansion of the product is

$$1-\frac{1}{2^s}-\frac{1}{3^s}-\frac{1}{5^s}-\frac{1}{7^s}-\frac{1}{11^s}-\cdots$$

But there's more! We can also pick two not-1 terms and all the rest 1's:

$$\begin{aligned}\frac{1}{\zeta(s)}&=\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{11^s}\right)\left(1-\frac{1}{13^s}\right)\cdots\\&=1-\left(\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\cdots\right)\\&\quad+\left(\frac{1}{6^s}+\frac{1}{10^s}+\frac{1}{14^s}+\frac{1}{15^s}+\frac{1}{21^s}+\cdots\right)+\cdots\end{aligned}$$

Then add in all possible plucks of 3 not-1's:

$$\begin{aligned}\frac{1}{\zeta(s)}&=\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{11^s}\right)\left(1-\frac{1}{13^s}\right)\cdots\\&=1-\left(\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\cdots\right)\\&\quad+\left(\frac{1}{6^s}+\frac{1}{10^s}+\frac{1}{14^s}+\frac{1}{15^s}+\frac{1}{11^s}+\cdots\right)+\\&\quad-\left(\frac{1}{30^s}+\frac{1}{42^s}+\frac{1}{66^s}+\frac{1}{70^s}+\frac{1}{78^s}+\cdots\right)\end{aligned}$$

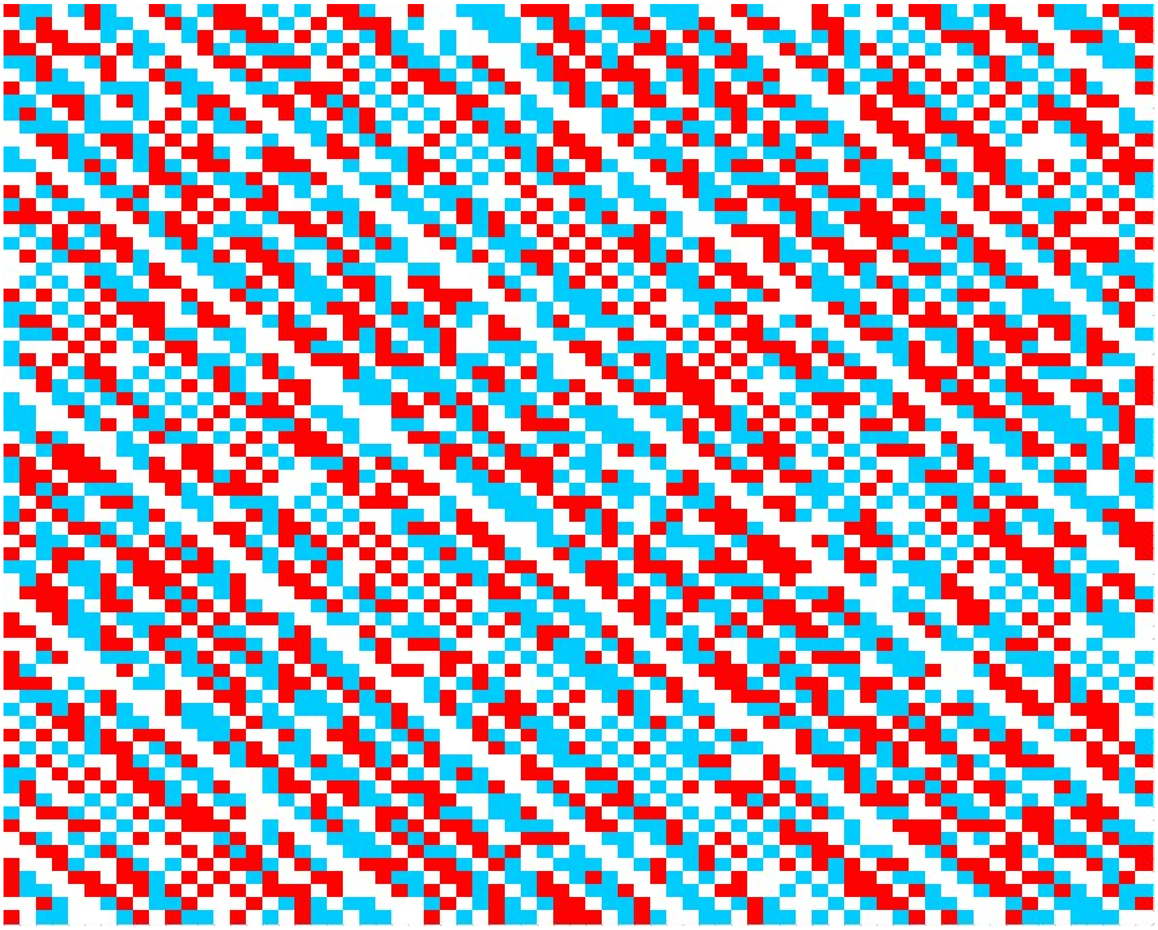
Continuing this process leads to

$$\frac{1}{\zeta(s)}=1-\frac{1}{2^s}-\frac{1}{3^s}-\frac{1}{5^s}+\frac{1}{6^s}-\frac{1}{7^s}+\frac{1}{10^s}-\frac{1}{11^s}-\frac{1}{13^s}+\frac{1}{14^s}+\frac{1}{15^s}-\frac{1}{17^s}-\cdots$$

The Mobius Function

$$\mu(n)=\begin{cases}0 & \text{if} \quad n \text{ is not squarefree}\\-1 & \text{if} \quad n \text{ has an odd number of prime factors}\\1 & \text{if} \quad n \text{ has an even number of prime factors}\end{cases}$$

$$\frac{1}{\zeta(s)}=\sum_n\frac{\mu(n)}{n^s}$$



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