# Convergence of a Catalan Series

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The well known Catalan numbers $C_n$ are named after Belgian mathematician Eugene Charles Catalan (1814–1894), who found them in his investigation of well-formed sequences of left and right parentheses. As Martin Gardner (1914–2010) wrote in *Scientific American* [**2**], they have the propensity to "pop up in numerous and quite unexpected places." They occur, for example, in the study of triangulations of convex polygons, planted trivalent binary trees, and the moves of a rook on a chessboard [**1, 2, 3, 4, 6**].

The Catalan numbers $C_n$ are often defined by the explicit formula $C_n = \frac{1}{n+1}\binom{2n}{n}$, where $n \geq 0$ [**1, 4, 6**]. Since $(n+1) \mid \binom{2n}{n}$, it follows that every Catalan number is a positive integer. The first five Catalan numbers are 1, 1, 2, 5, and 14. Catalan numbers can also be defined by the recurrence relation $C_{n+1} = \frac{4n+2}{n+2} C_n$, where $C_0 = 1$. So $\lim\limits_{n\to\infty} \frac{C_{n+1}}{C_n} = 4$.

Here we study the convergence of the series $\sum_{n=0}^{\infty} \frac{1}{C_n}$ and evaluate the sum. Since $\lim\limits_{n\to\infty} \frac{C_{n+1}}{C_n} = 4$, the ratio test implies that the series $\sum_{n=0}^{\infty} \frac{x^n}{C_n}$ converges for $|x| < 4$. Consequently, the series $\sum_{n=0}^{\infty} \frac{1}{C_n}$ converges. We evaluate this infinite sum using generating functions, plus fundamental tools from the differential and integral calculus.

## Sum of the series $\sum\limits_{n=0}^{\infty} \frac{1}{C_n}$

To this end, let $f(x)$ be the generating function of the reciprocals of Catalan numbers, $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{C_n}$. We compute the sum in three steps. First, we find an ordinary dif-

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ferential equation satisfied by $f(x)$, then, after solving the differential equation in the interval $(0, 4)$, we compute $f(1)$.

We first rewrite $f(x)$ as $f(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{C_n}$. So

$$f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{C_n} = \sum_{n=0}^{\infty} \frac{n+1}{C_{n+1}} x^n.$$

Since $\frac{n+2}{C_n} = \frac{4n+2}{C_{n+1}}$, by the recurrence relation, this yields

$$\sum_{n=0}^{\infty} \frac{n+2}{C_n} x^n = \sum_{n=0}^{\infty} \frac{4n+2}{C_{n+1}} x^n,$$

$$\sum_{n=0}^{\infty} \frac{n}{C_n} x^n + 2\sum_{n=0}^{\infty} \frac{x^n}{C_n} = \sum_{n=0}^{\infty} \frac{4(n+1)}{C_{n+1}} x^n - 2\sum_{n=0}^{\infty} \frac{x^n}{C_{n+1}},$$

$$xf'(x) + 2f(x) = 4f'(x) - \frac{2}{x}[f(x) - 1],$$

$$x(x-4)f'(x) + (2x+2)f(x) = 2. \tag{1}$$

This is a first-order differential equation for $f(x)$ with the initial conditions $f(0) = 1 = f'(0)$.

To facilitate solving (1) for $x \neq 0$, we introduce the function $g(x) = \left|\frac{4-x}{x}\right|^{3/2}$. Then $\frac{g'(x)}{g(x)} = \frac{-6}{x(4-x)}$. This implies that

$$[x(x-4)g(x)]' = (2x+2)g(x). \tag{2}$$

Multiplying (1) by $g(x)$, we get

$$x(x-4)f'(x)g(x) + (2x+2)f(x)g(x) = 2g(x).$$

Using (2), this can be rewritten as

$$[x(x-4)f(x)g(x)]' = 2g(x).$$

But, again using (2),

$$\begin{aligned} \{x(x-4)[f(x)-1]g(x)\}' &= [x(x-4)f(x)g(x)]' - [x(x-4)g(x)]' \\ &= 2g(x) - (2x+2)g(x) \\ &= -2xg(x), \end{aligned}$$

consequently,

$$x(x-4)[f(x)-1]g(x) = -2\int xg(x)\,dx + C_1$$

$$f(x) = 1 + \frac{2\int xg(x)\,dx - C_1}{x(4-x)g(x)},$$

where $C_1$ is a constant.

Suppose $0 < x < 4$. Then

$$\int xg(x)\,dx = \int x\left(\frac{4-x}{x}\right)^{3/2}\,dx$$
$$= \int \frac{(4-x)^{3/2}}{x^{1/2}}\,dx.$$

Letting $x = u^2$, this implies that

$$\int xg(x)\,dx = 2\int (4-u^2)^{3/2}\,du$$
$$= \frac{1}{2}u(4-u^2)^{3/2} + 3u(4-u^2)^{1/2} + 12\arcsin\frac{u}{2} + C_2$$
$$= \frac{1}{2}\sqrt{x}(4-x)^{3/2} + 3\sqrt{x}(4-x)^{1/2} + 12\arcsin\frac{\sqrt{x}}{2} + C_2,$$

where $C_2$ is another constant. Therefore, we have

$$f(x) = 1 + \frac{\sqrt{x}(4-x)^{3/2} + 6\sqrt{x}(4-x)^{1/2} + 24\arcsin\frac{\sqrt{x}}{2} + 2C_2 - C_1}{x(4-x)\left(\frac{4-x}{x}\right)^{3/2}}$$
$$= 1 + \frac{x(4-x)^{3/2} + 6x(4-x)^{1/2} + 24\sqrt{x}\arcsin\frac{\sqrt{x}}{2} + C\sqrt{x}}{(4-x)^{5/2}},$$

where $C = 2C_2 - C_1$. Since $f(0) = 1 = f'(0)$, $C = 0$. Thus

$$f(x) = 1 + \frac{x(4-x)^{3/2} + 6x(4-x)^{1/2} + 24\sqrt{x}\arcsin\frac{\sqrt{x}}{2}}{(4-x)^{5/2}} = \sum_{n=0}^{\infty}\frac{x^n}{C_n}.$$

When $x = 1$, this yields

$$\sum_{n=0}^{\infty}\frac{1}{C_n} = 1 + \frac{3^{3/2} + 6\cdot 3^{1/2} + 24\arcsin 1/2}{3^{5/2}}$$
$$= 2 + \frac{4\sqrt{3}}{27}\pi. \qquad (3)$$

Thus, the series $\sum_{n=0}^{\infty}\frac{1}{C_n}$ converges to the limit $2 + \frac{4\sqrt{3}}{27}\pi$, which is approximately 2.80613305077. Note that already $\sum_{n=0}^{22}\frac{1}{C_n} \approx 2.80613305077$; so the series converges to the limit remarkably fast.

## Additional consequences of the differential equation

Letting $x = 1$ in (1), we get

$$\begin{aligned} 3f'(1) &= 4f(1) - 2 \\ &= 4\left(2 + \frac{4\sqrt{3}}{27}\pi\right) - 2 \\ f'(1) &= 2 + \frac{16\sqrt{3}}{81}\pi. \end{aligned}$$

Since $f'(x) = \sum_{n=0}^{\infty} \frac{n+1}{C_{n+1}} x^n$, it follows that

$$\sum_{n=0}^{\infty} \frac{n+1}{C_{n+1}} = 2 + \frac{16\sqrt{3}}{81}\pi.$$

Since the differential equation is infinitely differentiable, it follows from (1) that

$$x(x-4)f''(x) + (4x-2)f'(x) + 2f(x) = 0. \tag{4}$$

This yields $3f''(1) = 2f'(1) + 2f(1)$, so $f''(1) = \frac{8}{3} + \frac{56\sqrt{3}}{243}\pi$. Since

$$f''(x) = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{C_{n+2}} x^n,$$

this implies that

$$\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{C_{n+2}} = \frac{8}{3} + \frac{56\sqrt{3}}{243}\pi.$$

Differentiating (4) with respect to $x$, it follows similarly that $f'''(1) = 2f'(1)$, so

$$\sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{C_{n+3}} = 4 + \frac{32\sqrt{3}}{81}\pi.$$

Clearly, this technique can be employed to evaluate further sums of the form

$$\sum_{n=0}^{\infty} \frac{(n+1)(n+2)\cdots(n+k)}{C_{n+k}},$$

where $k \geq 1$.

## Sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{C_n}$

We now turn to to solving (1) for $-4 < x < 0$. We have

$$\begin{aligned}\int xg(x)\,dx &= \int x\left|\frac{4-x}{x}\right|^{3/2} dx \\ &= -\int \frac{(4-x)^{3/2}}{\sqrt{x}}\,dx.\end{aligned}$$

Letting $x = -u^2$, this gives

$$\begin{aligned}\int xg(x)\,dx &= -\int \frac{(u^2+4)^{3/2}}{u}(-2u)\,du \\ &= 2\int (u^2+4)^{3/2}\,du \\ &= 2\left[\frac{u(u^2+4)^{3/2}}{4} + \frac{3}{2}u\sqrt{u^2+4} + 6\ln\left|u+\sqrt{u^2+4}\right|\right] + C_3 \\ &= \frac{1}{2}u(u^2+4)^{3/2} + 3u\sqrt{u^2+4} + 12\ln\left|u+\sqrt{u^2+4}\right| + C_3 \\ &= \frac{1}{2}\sqrt{|x|}(4-x)^{3/2} + 3\sqrt{|x|(4-x)} + 12\ln\left(\sqrt{|x|}+\sqrt{|4-x|}\right) + C_3,\end{aligned}$$

where $C_3$ is a constant.

Consequently, as before, we have

$$\begin{aligned}f(x) &= 1 + \frac{\sqrt{|x|}(4-x)^{3/2} + 6\sqrt{|x|(4-x)} + 24\ln\left(\frac{\sqrt{|x|}+\sqrt{|4-x|}}{C_4}\right)}{x(4-x)\left|\frac{4-x}{x}\right|^{3/2}} \\ &= 1 - \frac{|x|(4-x)^{3/2} + 6\sqrt{|x|(4-x)} + 24\sqrt{|x|}\ln\left(\frac{\sqrt{-x}+\sqrt{4-x}}{C_4}\right)}{(4-x)^{5/2}}.\end{aligned}$$

where $C_4$ is a nonzero constant. Since $f(0) = 1 = f'(0)$, $C_4 = 2$. Thus

$$f(x) = 1 - \frac{|x|(4-x)^{3/2} + 6\sqrt{|x|(4-x)} + 24\sqrt{|x|}\ln\left(\frac{\sqrt{-x}+\sqrt{4-x}}{2}\right)}{(4-x)^{5/2}}.$$

Since $f(-x)$ generates the alternating series $\sum_{n=0}^{\infty} \frac{(-x)^n}{C_n}$ for $0 < x < 4$, setting $x = 1$ we get

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n}{C_n} &= 1 - \frac{5^{3/2} + 6\cdot 5^{1/2} + 24\ln\phi}{5^{5/2}} \\ &= \frac{14}{25} - \frac{24\sqrt{5}}{125}\ln\phi, \qquad (5)\end{aligned}$$

where $\phi$ is the well-known *golden ratio* $\frac{1+\sqrt{5}}{2}$.

Formulas (3) and (5) can be employed to compute the sums of the subseries $\sum_{n=0}^{\infty} \frac{1}{C_{2n}}$ and $\sum_{n=0}^{\infty} \frac{1}{C_{2n+1}}$:

$$\sum_{n=0}^{\infty} \frac{1}{C_{2n}} = \frac{32}{25} + \frac{2\sqrt{3}}{27}\pi - \frac{12\sqrt{5}}{125} \ln \phi,$$

$$\sum_{n=0}^{\infty} \frac{1}{C_{2n+1}} = \frac{18}{25} + \frac{2\sqrt{3}}{27}\pi + \frac{12\sqrt{5}}{125} \ln \phi.$$

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**Summary.** This article studies the convergence of the infinite series of the reciprocals of the Catalan numbers. We extract the sum of the series as well as some related ones, illustrating the power of the calculus in the study of the Catalan numbers.

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