Math 2C Noonburg - Chapter 5 Project: A New Predator-prey Equation 5/21/15

1 Exposition

This problem is an excellent illustration of how scientists from different disciplines can work together to obtain interesting results. In a recently pulbished paper, written jointly by a mathematician and a biologist, the following predator-prey model is studied:

$$\begin{aligned} x' &= x(a - \lambda x) - yP(x) \\ y' &= -\delta y - \mu y^2 + cyP(x). \end{aligned} \tag{1}$$

The parameters a, λ, δ, μ, and c are all assumed to be positive.

This is very similar to the previous predator-prey models we have looked at, except for the interaction terms–$yP(x)$ and $cyP(x)$. In the paper, the biologist first gives a game-theoretic argument to show that if the prey are able to use group defense behavior, the predators' attack rate $P(x)$ may actually decrease when the prey population gets larger enough. The function $P(x)$ is a non-negative function of x formally referred to by population biologists as the predator's functional response, and in the paper it is shown that, if the prey's group behavior is taken into account, P will satisfy the conditions

1. $P(0) = 0$.
2. $P(x) \geq 0$, for $x > 0$.
3. there exists a positive value $x = x_c$, such that

$$P'(x) > 0 \texttt{ if } 0 \leq x \leq x_c, \texttt{ and } P'(x) \leq 0 \texttt{ for } x \geq x_c$$

The function chosen by the authors to represent P is a rational function of the form

$$P(x) = \frac{mx}{\alpha x^2 + \beta x + 1}$$

The general idea behind the paper is to show that with only these minimum assumptions on the predator's response function, the dynamics of the resulting system can be shown to be limited to a small number of possibilities.

It is first shown that if $K > \dfrac{c(\delta + a)^2}{4\lambda\delta}$, then all trajectories starting in the positive quadrant will ultimately enter the triangular region T of the (x, y)-plane, bounded by the lines $x = 0, y = 0$, and $y = K - cx$. Once inside, they cannot exit from the closed region T (that is, the region T with its boundary.)

The term $-\mu y^2$ in the second equation in (1) represents intraspecific competition among the predators. For example, this might be due to fighting over the food supply. If μ is small enough, it is shown (using a fair amount of algebra) that there can be at most two equilibrium points in the interior of T. Using dynamical systems theory, the mathematician is able to show that in the case where all the equilibrium points are hyperbolic (remember this means their Jacobian has no eigenvalues with zero real part), and no limit cycles exist, there are exactly four types of phase planes possible. A diagram showing them is included in the paper, and looking at this part of the paper should be very profitable. More mathematics is used to show that under certain further conditions there can be a limit cycle.

Write out complete answers to problems 1–8. Let $\mu = 0.005$ and let $P(x)$ be the rational function

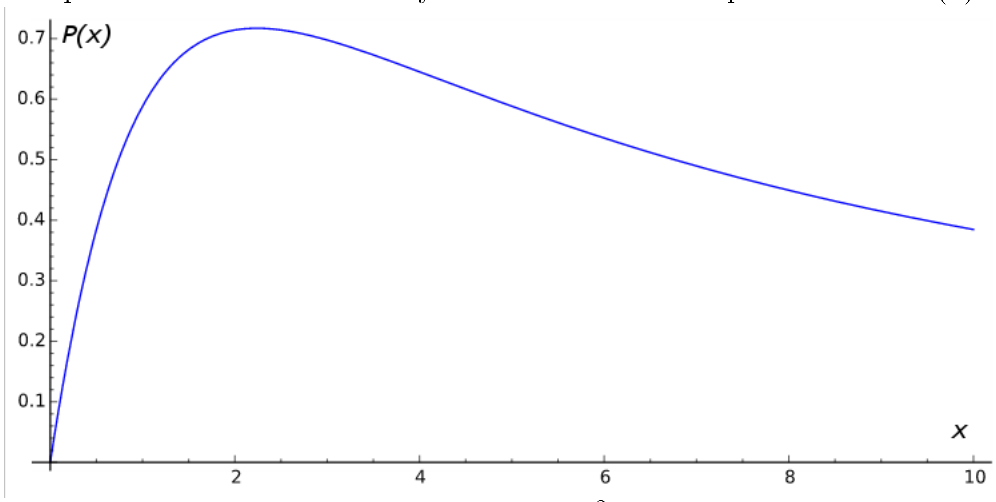
$$P(x) = \frac{x}{0.2x^2 + 0.5x + 1}$$

2 Problems

1. Check that $(0,0)$ and $\left(\frac{a}{\lambda},0\right)$ are the only equilibrium points of (1) where at least one of the populations has died out.

ANS: At equilibrium, $x' = x(a-\lambda x) - \dfrac{xy}{0.2x^2+0.5x+1} = 0$ and $y' = -\delta y - 0.005y^2 + \dfrac{cxy}{0.2x^2+0.5x+1} = 0$. If at least one one of the populations has died out then either $x=0$ or $y=0$ or both. If $x=0$ then $x'=0$ and $y' = -\delta y(1+y/200) = 0$ so, since $y \geq 0$ we must have $y=0$. If $y=0$, and $x \neq 0$ then $x' = 0 \Rightarrow x = \dfrac{a}{\lambda}$.

2. The plot shown below is certainly consistent with the requirements for $P(x)$.



Clearly, $P(0)=0$. Also, $P'(x) = \dfrac{1-0.2x^2}{(0.2x^2+0.5x+1)^2}$ shows P is increasing on $(0,\sqrt{5})$ and decreasing thereafter.

3. Show that the Jacobian matrix for (1) is

$$J(x,y) = \begin{pmatrix} a-2\lambda x - yP'(x) & -P(x) \\ cyP'(x) & -\delta - 2\mu y + cP(x) \end{pmatrix}$$

and use it to show that $(0,0)$ is always a saddle point of the system, and $\left(\frac{a}{\lambda},0\right)$ can be either a sink or a saddle point.

ANS: It's fairly simple that these are the partial derivatives of the Jacobian $\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$. $J(0,0) = \begin{pmatrix} a & 0 \\ 0 & -\delta \end{pmatrix}$ has $\texttt{det}(J(0,0)) = -a\delta < 0$, so that's a saddle. Also, $J\left(\frac{a}{\lambda},0\right) = \begin{pmatrix} -a & -P\left(\frac{a}{\lambda}\right) \\ 0 & -\delta + cP\left(\frac{a}{\lambda}\right) \end{pmatrix} = \begin{pmatrix} -a & \dfrac{-5a\lambda}{(4a+5\lambda)^2+55\lambda^2} \\ 0 & -\delta + \dfrac{5ac\lambda}{(4a+5\lambda)^2+55\lambda^2} \end{pmatrix}$ has $\texttt{det}\left(J\left(\frac{a}{\lambda},0\right)\right) = a\left(\delta - cP\left(\frac{a}{\lambda}\right)\right) = a\delta - \dfrac{5a^2c\lambda}{(4a+5\lambda)^2+55\lambda^2}$, which could be positive or negative, depending on the values of the parameters, a, δ, c and λ. The $\texttt{tr}\left(J\left(\frac{a}{\lambda},0\right)\right) = cP\left(\frac{a}{\lambda}\right) - (a+\delta)$ has a maximum where $x = \dfrac{a}{\lambda} = \sqrt{5}$ at which point $\texttt{tr}\left(J\left(\sqrt{5},0\right)\right) = \dfrac{c(8\sqrt{5}-10)}{11} - (a+\delta)$ could be positive or negative...
Ok, let's set about relating the trace directly to the determinant. We have

$$\texttt{tr}(J) = cP\left(\frac{a}{\lambda}\right) - (a+\delta)$$
$$\texttt{det}(J) = a\left(\delta - cP\left(\frac{a}{\lambda}\right)\right)$$

So $cP\left(\frac{a}{\lambda}\right) = a+\delta+\texttt{tr(J)}$. Substituting this into the second equation gives us $\texttt{det}(J) = a\left(\delta - a - \delta - \texttt{tr}(J)\right) = -a\left(a+\texttt{tr}(J)\right)$ which means that if the trace is positive, then the determinant is negative. Thus this equilibrium is either a saddle (if the trace is positive) or a sink (if the determinant is positive.)

4. Show that is $(\bar{x}, \bar{y})$ is any equilibrium point of (1) in the interior of the triangle T, it must satisfy the two equations

$$\bar{y} = \frac{cP(\bar{x}) - \delta}{\mu}$$
$$\bar{y} = \frac{\bar{x}(a - \lambda\bar{x})}{P(\bar{x})} = (a - \lambda\bar{x})(0.2\bar{x}^2 + 0.5\bar{x} + 1).$$

ANS: To be at equilibrium we require that $x' = 0$ and $y' = 0$. If $y \neq 0$ then $y' = 0 \Rightarrow \bar{y} = \dfrac{cP(\bar{x}) - \delta}{\mu}$. Also $x' = 0 \Rightarrow \bar{y} = \dfrac{\bar{x}(a - \lambda\bar{x})}{P(\bar{x})}$ and, after substituting the formula for $P(x)$ and reducing by a factor of x, $\bar{y} = (a - \lambda\bar{x})(0.2\bar{x}^2 + 0.5\bar{x} + 1)$

5. With $\mu = 0.005$, find positive values of a, λ, δ, and c such that the two curves defined in question 4 have no intersections; that is, no interior equilibrium points exist. Using these values of the parameters, draw a phase portrait and describe what happens to the predator and prey populations as $t \to \infty$. Use several initial conditions with $x(0)$ $y(0)$ both positive.
ANS: This is where things start getting interesting. Taking a big picture view, the first curve is like, well, a shifted Witch of Agnesi multiplied by x and then shifted around some more. Call it an odd shift-witch. We have an amplification factor of $200c$ and then a shift down by δ. The second curve is a cubic polynomial tending to $-\infty$ with a single real zero at $\dfrac{a}{\lambda}$. The zero of the "odd shift-witch" is where $P(x) = \dfrac{\delta}{c}$.
$P(x) = \dfrac{\delta}{c} \Leftrightarrow cx = \delta(0.2x^2 + 0.5x + 1)$
This is quadratic, so we can write it in standard form: $x^2 + \dfrac{5\delta - 10c}{2\delta}x + 5 = 0$ and solve for x:
$x - \dfrac{10c - 5\delta}{4\delta} = \sqrt{-5 + \left(\dfrac{5\delta - 10c}{4\delta}\right)^2} \Leftrightarrow x = \dfrac{10c - 5\delta}{4\delta} \pm \dfrac{\sqrt{(10c - 5\delta)^2 - 80\delta^2}}{4\delta}$. To choose parameter values that lead to no intersection of the curves in T, we want to choose a, λ, δ and c so that
$\dfrac{a}{\lambda} < \dfrac{10c - 5\delta}{4\delta} - \dfrac{\sqrt{100c^2 - 100\delta c - 55\delta^2}}{4\delta}$. To achieve this, you'd tend to want small a and δ and large λ and c.

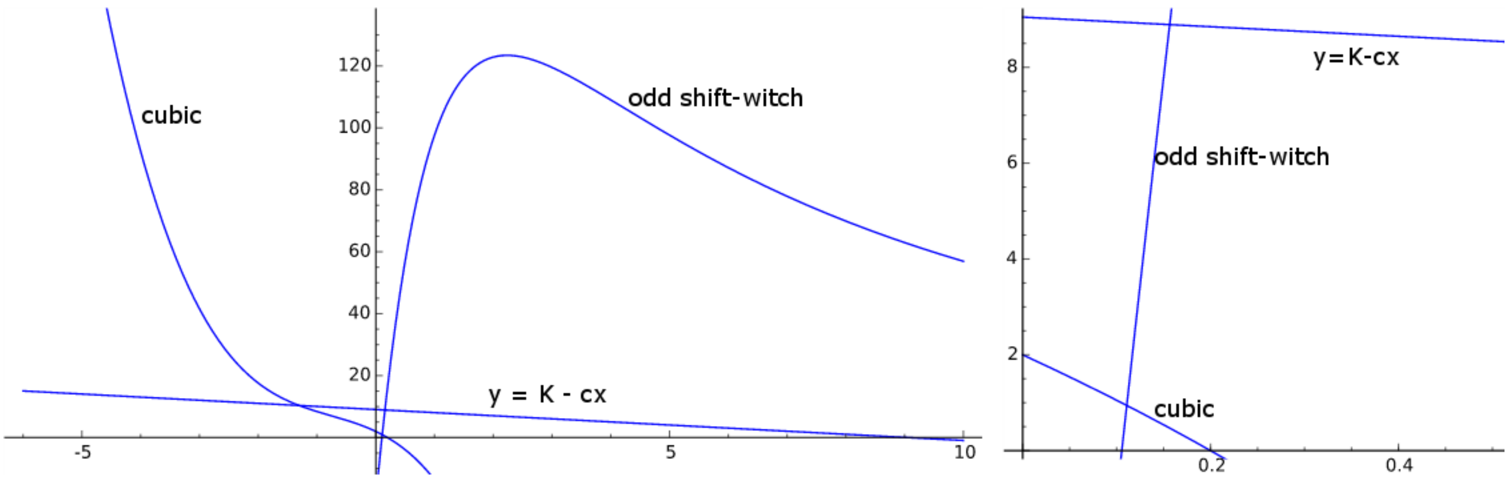
Let's do some Sage work. Here's an instance of the code I wrote to analyze this pred-prey situation:

```
x,a,el,delta,c,K,Kmin,v,oddzero,cubezero =
           var('x a el delta c K Kmin v oddzero cubezero')
a,el,c,delta = 2,10,1,0.1
Kmin=c*(delta+a)^2/(4*el*delta);Kmin
P=function('P',x)
P=0.2*x^2+0.5*x+1
v=P(a/el)
K=((c*v)-(a+delta))^2-4*a*(delta-c*v);K
y1=function('y1',x)
y2=function('y2',x)
y3=function('y3',x)
y1=(c*x/(0.2*x^2+0.5*x+1)-delta)/0.005
y2=(a-el*x)*(0.2*x^2+0.5*x+1)
y3=K-c*x
g=plot(y1,x,-6,10,ymin=-K,ymax=15*K)
g+=plot(y2,x,-6,10,ymin=-K,ymax=15*K)
g+=plot(y3,x,-6,10,ymin=-K,ymax=15*K)
oddzero=((10*c-5*delta)-sqrt(100*c*c-100*delta*c-55*delta^2))/(4*delta);oddzero
cubezero=a/el;cubezero
show(g)
```

I used "`el`" instead of "`lambda`" because it seems that "`lambda`" is a keyword and can't be a variable name.

There are a bunch of variables, but only the key parameters are initialized here, with `a,el,c,delta = 2,10,1,0.1` making the odd shift-witch $y = 200(P(x) - 0.1)$ and the cubic, $y = (2 - 10x)(0.2x^2 + 0.5x + 1)$. We also see the the upper-bound of T: the line $y = K - cx$ where $K = (\mathtt{tr}(J))^2 - 4|J|$, which, with $x \approx \frac{a}{\lambda}$ is $(cv - (a + \delta))^2 - 4a(\delta - cv)$ where $v = P\left(\frac{a}{\lambda}\right)$ The output shows the two x-intercepts and the graph and the values of $K_{min} = \frac{c(\delta + a)^2}{4\delta\lambda}$, K and the zeros:

```
Kmin=1.10250000000000
K=9.04806400000000
oddzero=0.105497467698754
cubezero=1/5
```

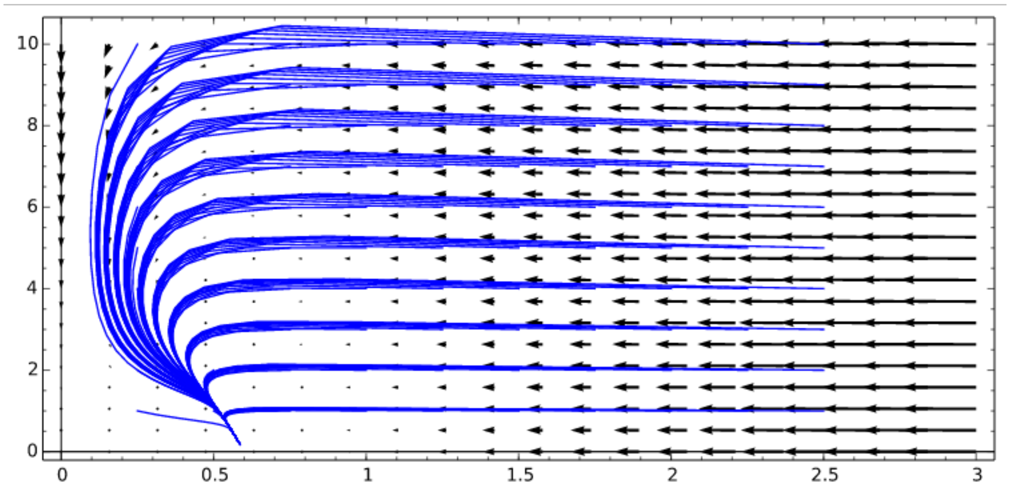


The second picture above shows a zoom (`g=plot(y1,x,0,1,ymin=0,ymax=K)`) to the point of intersection where there is a $\bar{y}$-value for the equilibrium at $\bar{x} = a/\lambda$.
Choosing `a,el,c,delta = 6,10,4,2` makes for a lack of intersection of the two curves.
Using these parameters, I construct the Sage instruction set:

```
x,y=var('x','y')
a,el,delta,c,K,Kmin,v,oddzero,cubezero = var('a el delta c K Kmin v oddzero cubezero')
a,el,c,delta = 6,10,4,2
f(x,y) = x*(a-el*x)-y*x/(0.2*x^2+0.5*x+1)
g(x,y) = -delta*y-y^2/200+c*y*x/(0.2*x^2+0.5*x+1)
VF = plot_vector_field([f(x, y), g(x, y)], [x, 0, 3], [y, 0, 10])
T = ode_solver()
T.function = lambda t, y: [f(y[0], y[1]), g(y[0], y[1])]

solutions = []
for x_0 in range(1,11):
    for y_0 in range(1,11):
        T.ode_solve(y_0 = [x_0/4, y_0], t_span = [0, 5], num_points = 50)
        solutions.append(line([p[1] for p in T.solution]))
show(sum(solutions) + VF)
```

to produce this phase portrait:



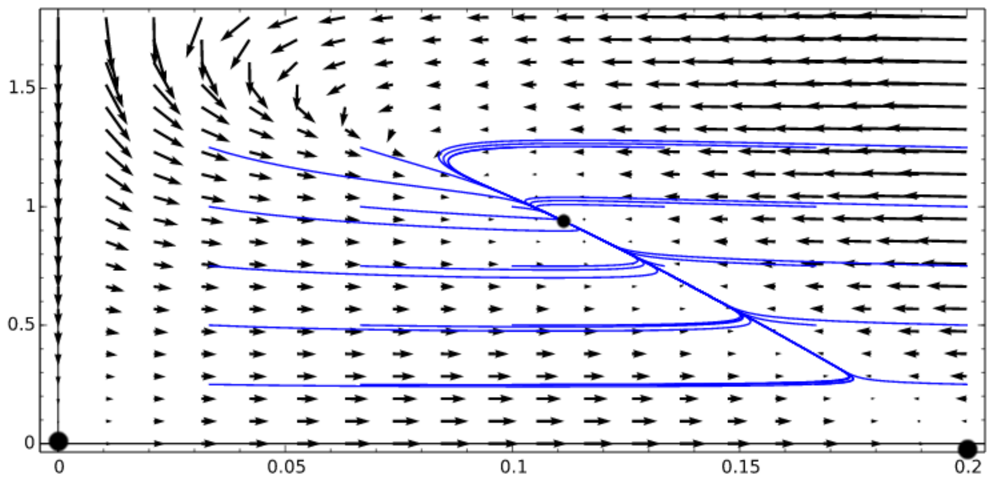
Showing that $(\bar{x}.\bar{y}) = \left(\frac{a}{\lambda}, 0\right)$ is a sink. To be sure, with $\lambda = 10$ and $a = 6$ the equilibrium at $(0.6, 0)$, $\texttt{det}(J) \approx 9.378$ and $\texttt{tr}(J) \approx -7.563$ is in the sink zone.

6. Find positive values of a, λ, δ, and c such that the two curves defined in part (4) have exactly one intersection. Use the Jacobian to determine the type of each of the equilibrium solutions. Is the equilibrium $\left(\frac{a}{\lambda}, 0\right)$ a sink or a saddle point? Draw the phase portrait and describe what happens to to the two populations as $t \to \infty$.
ANS: It's easy to construct a plot showing that there is exactly one point of intersection for the curves in part (4) with these parameter values: `a,el,c,delta = 2,10,1,0.1`. Here're the commands I used to get a phase portrait with these parameters (note I snuck in a function for $P(x)$:

```
x,y=var('x','y')
a,el,delta,c,K,Kmin,v,oddzero,cubezero = var('a el delta c K Kmin v oddzero cubezero')
a,el,c,delta = 2,10,1,0.1
P=function('P',x)
P=0.2*x^2+0.5*x+1
pp=function('pp',x)
pp=x/P(x)
v=P(a/el)
K=((c*v)-(a+delta))^2-4*a*(delta-c*v);K
f(x,y) = x*(a-el*x)-y*x/(0.2*x^2+0.5*x+1)
g(x,y) = -delta*y-y^2/200+c*y*x/(0.2*x^2+0.5*x+1)
VF = plot_vector_field([f(x, y), g(x, y)], [x, 0, 0.2], [y, 0, 1.8])
T = ode_solver()
T.function = lambda t, y: [f(y[0], y[1]), g(y[0], y[1])]

solutions = []
for x_0 in range(1,7):
  for y_0 in range(1,6):
    T.ode_solve(y_0 = [x_0/30, y_0/4], t_span = [0, 25], num_points = 100)
        solutions.append(line([p[1] for p in T.solution]))
show(sum(solutions) + VF)
```

As you can see, there is a coexistence equilibrium:



First we need to find the x-coordinate where the curves of problem (4) intersect. I used Sage's `find_root` command:

```
R.<x> = QQ[]
f=200*(x/(0.2*x^2+0.5*x+1)-0.1)-(2-5*x)*(0.2*x^2+0.5*x+1)
xbar = find_root(f,0,.3)
show(xbar)
ybar=P(xbar)
show(ybar)
Pder=pp.derivative(x);#Pder(xbar)
J=[[a-2*el*xbar-ybar*Pder(xbar), -pp(xbar)],[c*ybar*Pder(xbar),-delta-ybar/100+c*pp(xbar)]];show(J
```

Which produced $\bar{x} = 0.113987138787$ at which point $\bar{y} = 1.05959218295553$. And $J(\bar{x}, \bar{y}) \approx \begin{pmatrix} -1.22105 & -0.107576 \\ 0.941307 & -0.0030195 \end{pmatrix}$ whose trace is ≈ -1.22407 and whose determinant is ≈ 0.10495 Since $(\mathtt{tr}(J) \approx 1.5 > 4|J| \approx 0.408$, this equilibrium is a sink.

The equilibrium at $(0,0)$ is a saddle and the equilibrium at $(0.2,0)$ has $|J| = 0$.

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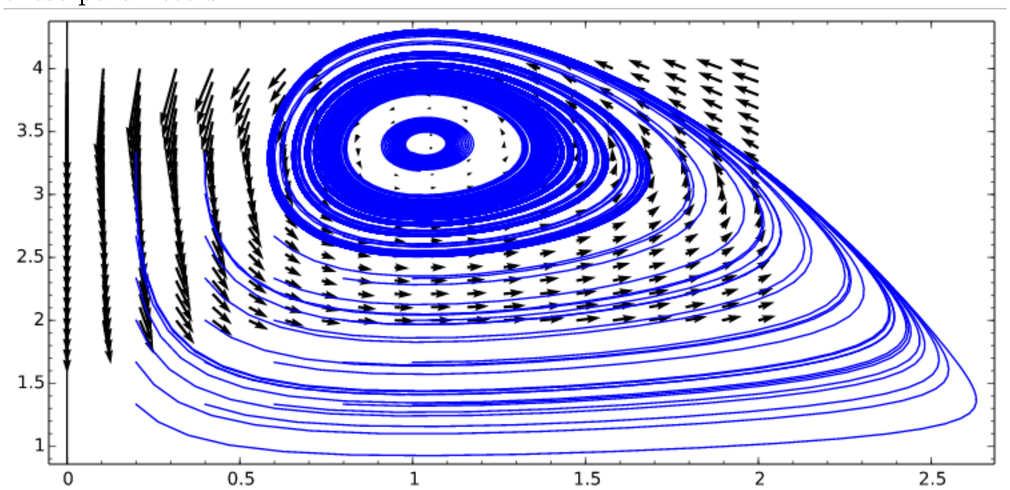
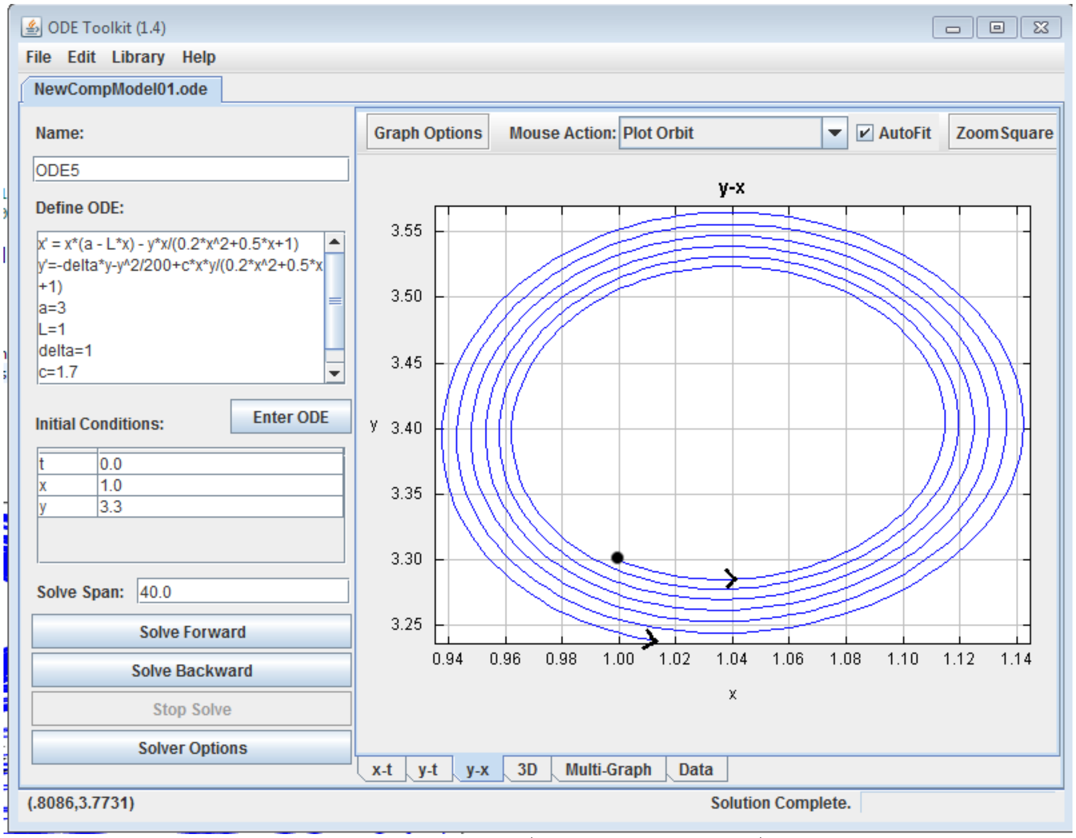
Alternatively, we can choose (as one team did) $a = 3, \lambda = 1, \delta = 1$, and $c = 1.7$. Here is a phase portrait with these parameters:



A limiting cycle perhaps?

The point of intersection is $(\bar{x}, \bar{y}) \approx (1.03722, 3.403019)$

Using ODEToolkit, you can pick an initial point close to this equilibrium and see that the curve is cycling out, just as you'd expect from a cyclic source:

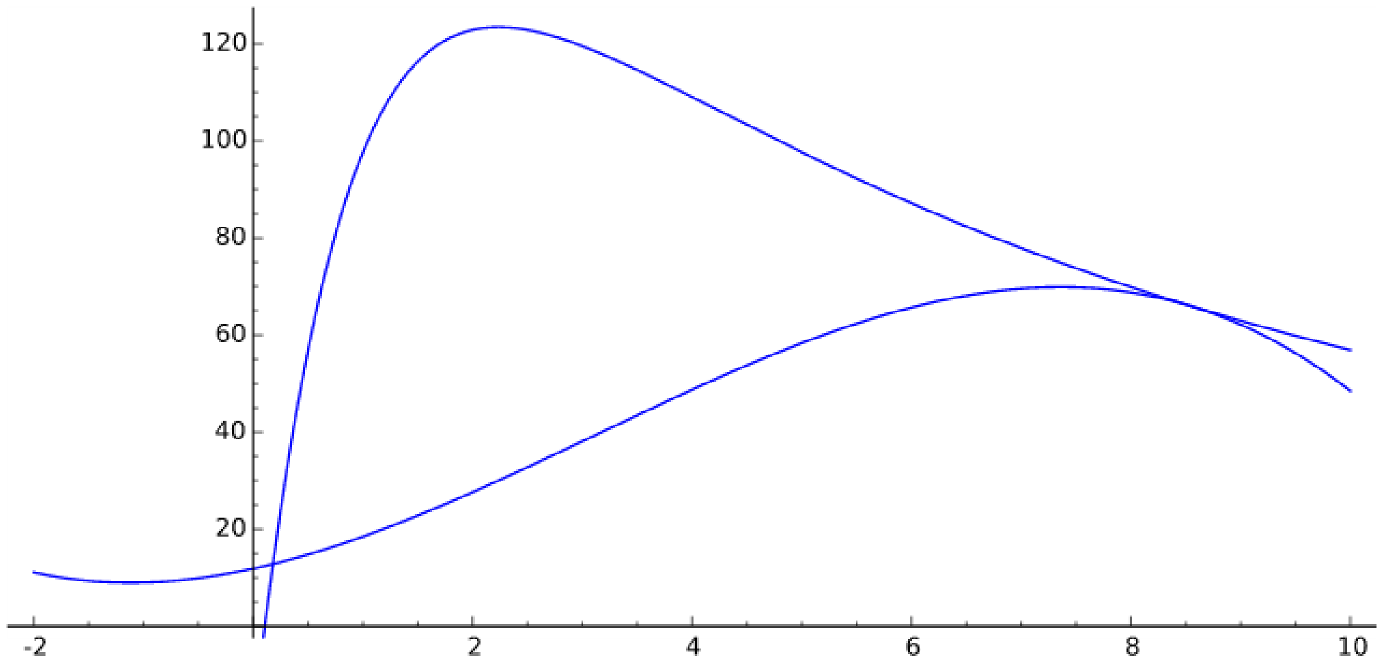


In fact, $J(\bar{x},\bar{y}) \approx J(1.03722, 3.40602) \approx \begin{pmatrix} 0.0371 & -0.5982 \\ 1.5104 & -0.01702 \end{pmatrix}$ so that $\mathtt{det}(J) \approx 0.903$ and $\mathtt{tr}(J) = 0.020$. a cyclic source.

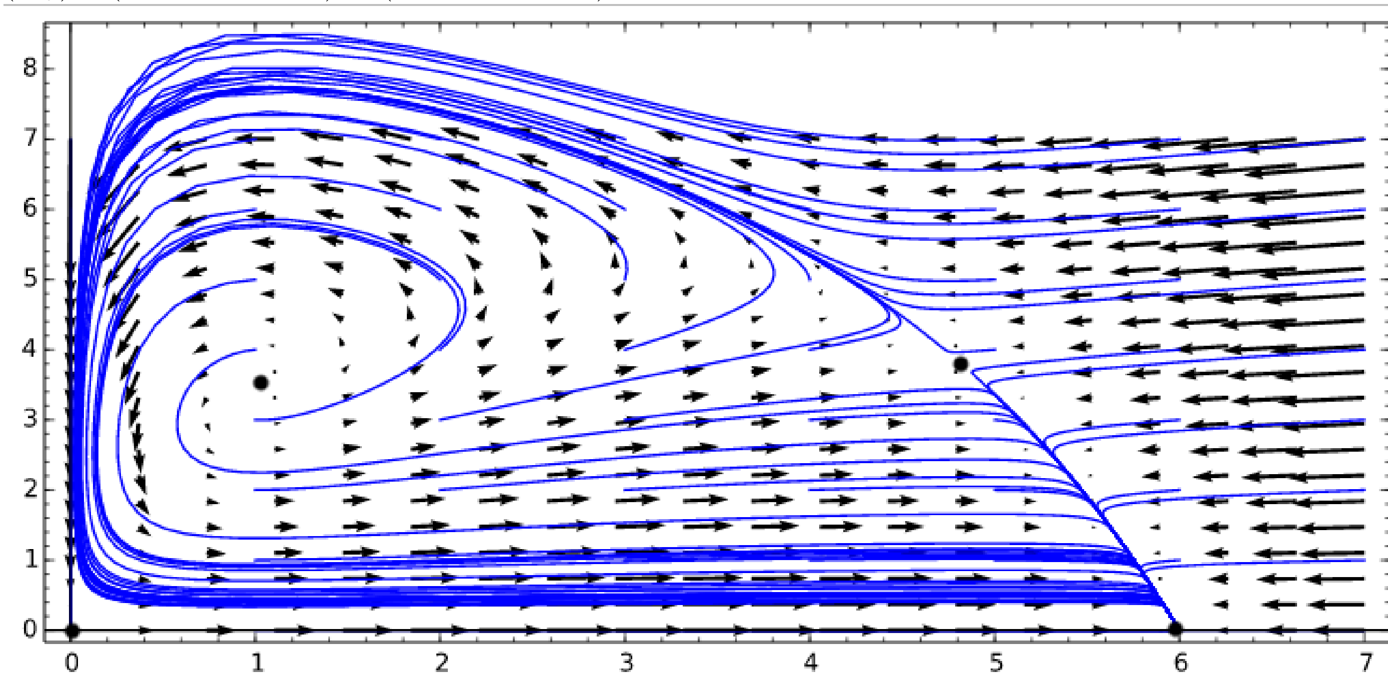
7. Find positive values of a, λ, δ, and c such that the two curves defined in part (4) have exactly two intersections. Use the Jacobian to determine the type of each of the equilibrium points (in this case you should have either a source and a saddle, or a sink and a saddle.) Draw the phase portrait and describe what happens to trajectories in each region of the positive quadrant.
ANS: Ok, folks, I'm not sure yet how to set parameters for a system with "exactly" two coexisting equilibria, but, by trial and error, I can get pretty close. Setting $\lambda = 1, c = 1$ and $\delta = 0.1$ I tried different values of a until I got one where the cubic crosses the odd shifted-witch once and then barely glances off it at a near-tangency point:

```
x,a,el,delta,c,K,Kmin,v,oddzero,cubezero = var('x a el delta c K Kmin v oddzero cubezero')
a,el,c,delta = 11.8647,1,1,0.1
Kmin=c*(delta+a)^2/(4*el*delta);Kmin
P=function('P',x)
P=0.2*x^2+0.5*x+1
v=P(a/el)
K=((c*v)-(a+delta))^2-4*a*(delta-c*v);K
y1=function('y1',x)
y2=function('y2',x)
y3=function('y3',x)
y1=(c*x/(0.2*x^2+0.5*x+1)-delta)/0.005
y2=(a-el*x)*(0.2*x^2+0.5*x+1)
y3=K-c*x
g=plot(y1,x,-2,10,ymin=0,ymax=125)
g+=plot(y2,x,-2,10,ymin=0,ymax=125)
g+=plot(y3,x,-2,10,ymin=0,ymax=125)
oddzero=((10*c-5*delta)-sqrt(100*c*c-100*delta*c-55*delta^2))/(4*delta);oddzero
cubezero=a/el;cubezero
show(g)
```

Produces this plot:



A more straight-forward approach whould have the cubic just cut across the arch of the odd shift-witch in the first quadrant by choosing, say $a = 2.4, \lambda = 0.4, \delta = 1$, and $c = 1.7$. The two points of intersection are now where $(\bar{x}, \bar{y}) = (1.03766, 3.44224)$ or $(4.79721, 3.84952)$. Here is a phase portrait of the solution:



The equilibrium at $(6, 0)$ is a sink: $J(6, 0) \approx \begin{pmatrix} -6 & -30/61 \\ 0 & -10/61 \end{pmatrix}$ So that $\texttt{det}(J) \approx 0.9836$ and $\texttt{tr}(J) \approx -6.1369$.

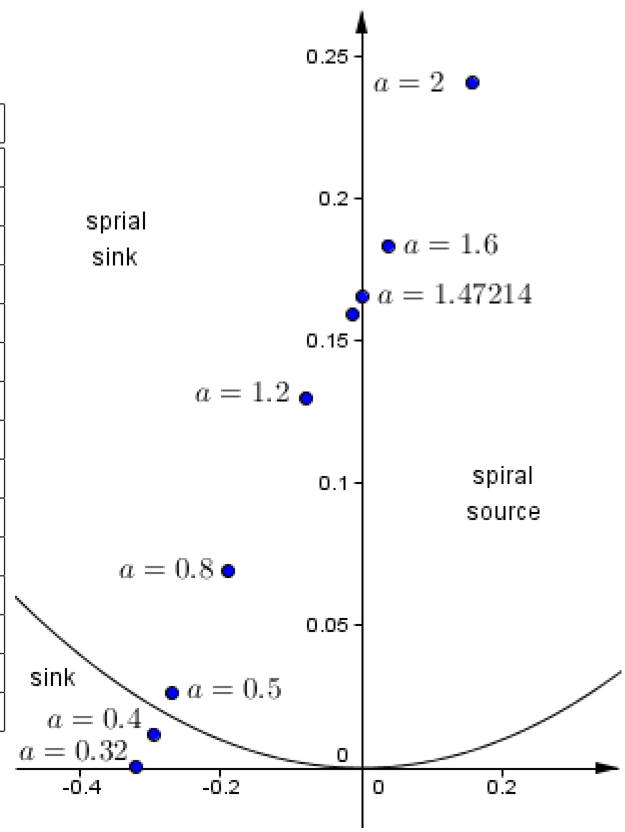
The equilibrium at $(1.03766, 3.44224)$ is a spiral source: $J(1.03766, 3.44224) \approx \begin{pmatrix} 0.6718 & -0.5984 \\ 1.5268 & -0.01721 \end{pmatrix}$ so that $\texttt{det}(J) \approx 0.9021$ and $\texttt{tr}(J) \approx 0.6546$

The equilibrium at $(4.79721, 3.84952)$ is a saddle: $J(4.79721, 3.84952) \approx \begin{pmatrix} -1.2211 & -0.600 \\ -0.3683 & -0.01925 \end{pmatrix}$ so that $\texttt{det}(J) \approx -0.244$

8. Let $\lambda = 0.6, \delta = 0.2, c = 0.5$ and the same value of $\mu = 0.005$ as before. For values of a between 0.32 and 2.0, there is a single interior equilibrium point C. Using the Jacobian, show that C changes from a spiral sink to a spiral source at $a = \bar{a} \approx 1.472$, so that a Hopf bifurcation occurs at this value of a and a limit cycle exists when $a > \bar{a}$. Draw phase portraits for $a = 0.5$ and $a = 2$, and for both cases describe what happens to the two populations in terms that a biologist might use.
ANS: The equilibrium must satisfy $100P(\bar{x}) - 40 = (a - 0.2\bar{x})(0.2\bar{x}^2 + 0.5\bar{x} + 1)$ which is a fifth degree polynomial in $\bar{x}$ that we want to show has only one positive zero if $0.32 < a < 2$. Hmmm. This implicity defines a surface in the $\bar{x}, a$ domain, but that seems like a lot of trouble to analyze. How about just making a table of values for $(\mathtt{tr}(J), \mathtt{det}(J))$ values over the range of a?
For better or worse, this is the code I'm using to produce these values:

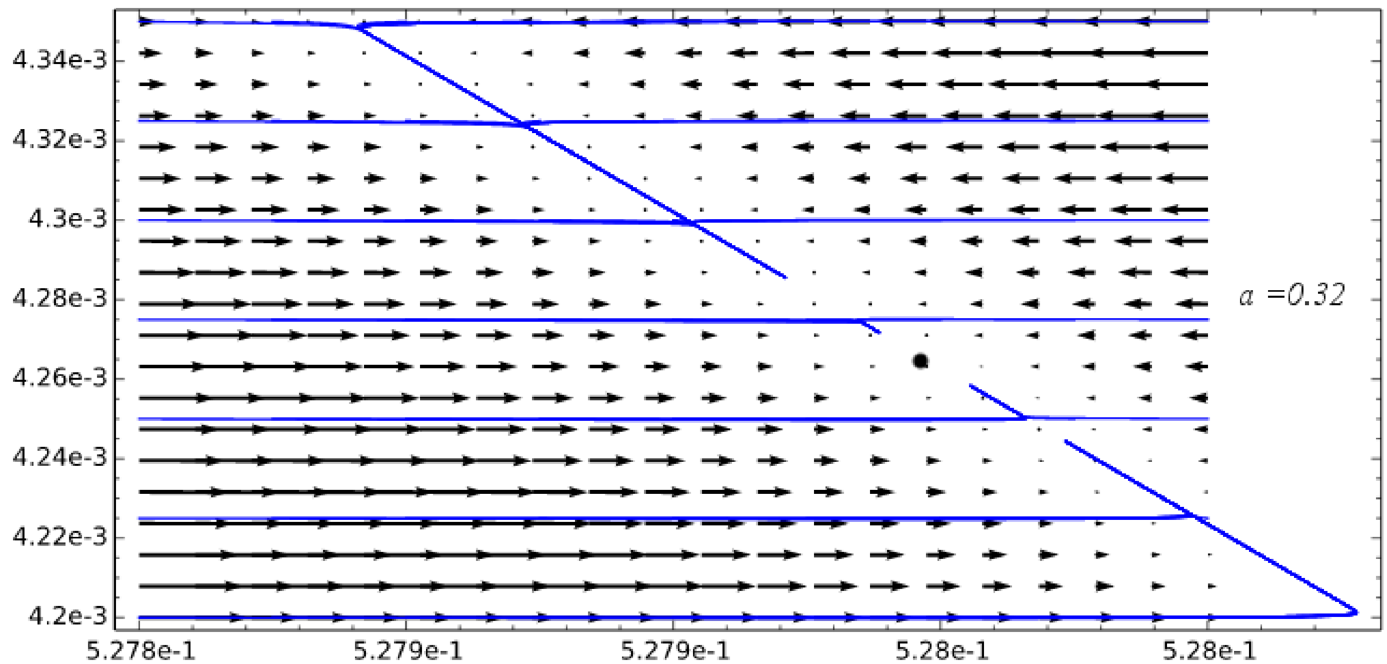
```
R.<x> = QQ[]
a,el,delta,c = var('a el delta c')
a,el,c,delta = 1.436,0.6,0.5,0.2
P=function('P',x)
P=0.2*x^2+0.5*x+1
pp=function('pp',x)
pp=x/P(x)
f=200*(c*x/(0.2*x^2+0.5*x+1)-delta)-(a-el*x)*(0.2*x^2+0.5*x+1)
xbar = find_root(f,0,4)
show(xbar)
ybar=(a-el*xbar)*P(xbar)
show(ybar)
Pder=pp.derivative(x);#Pder(xbar)
J=[[a-2*el*xbar-ybar*Pder(xbar), -pp(xbar)],[c*ybar*Pder(xbar),-delta-ybar/100+c*pp(xbar)]];show(J)
J2=[[0, -pp(0.2)],[0,-delta+c*pp(0.2)]];show(J2)
D=J[0][0]*J[1][1]-J[0][1]*J[1][0];D
T=J[0][0]+J[1][1];T
K=T^2-4*D;K
```

a	$\bar{x}$	$\bar{y}$	$\mathtt{det}(J)$	$\mathtt{tr}(J)$	K
3.2	0.5279	0.004268	0.00047	-0.36159	0.09
0.4	0.5299	0.10843	0.01192	-0.2950	0.04
0.5	0.53228	0.235893	0.02324	-0.26875	-0.033
0.6	0.5347	0.3698	0.04055	-0.2422	-0.10
0.8	0.5396	0.63245	0.069196	-0.1884	-0.24
1	0.5446	0.8965	0.097832	-0.1336	-0.37
1.2	0.5496	1.1620	0.1264	-0.07771	-0.5000
1.4	0.5547	1.4288	0.1550	-0.0208	-0.6200
1.46	0.5562	1.5092	0.1636	-0.00351	-0.6545
1.47	0.5565	1.5226	0.1650	-0.00062	-0.6602
1.47214	0.5565	1.52546	0.1654	1.1e-8	-0.6614
1.5	0.5572	1.5628	0.1693	0.00807	-0.6773
1.6	0.5598	1.6972	0.1836	0.0372	-0.7330
1.8	0.5650	1.9670	0.2121	0.0963	-0.8392
2	0.5703	2.2384	0.2406	0.1565	-0.9378

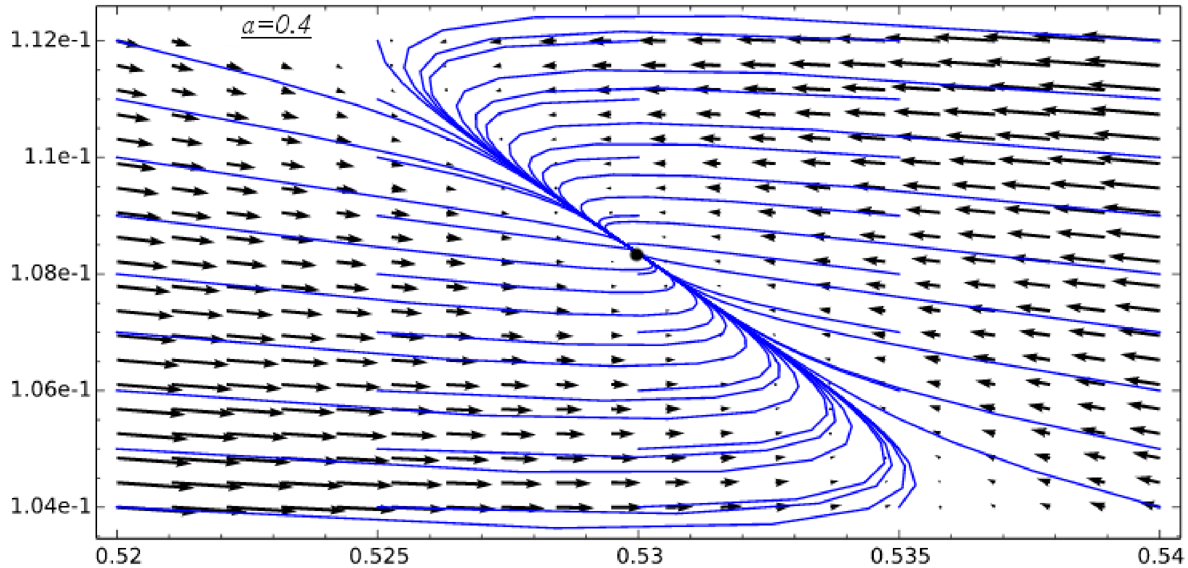


If $a = 0.32$ then equilibrium is a sink with (tr,det)$\approx (-0.36, 0.00047)$ The sink has det-trace values very close to the saddle region. Zooming way in for a phase portrait around this sink, as shown below, we still see very abrumpt transition from left/right motion to snapping to the eigenline and approaching the sink near $(0.528, 0.000427)$, however glacially.
The engine of the scaling/zoom code:

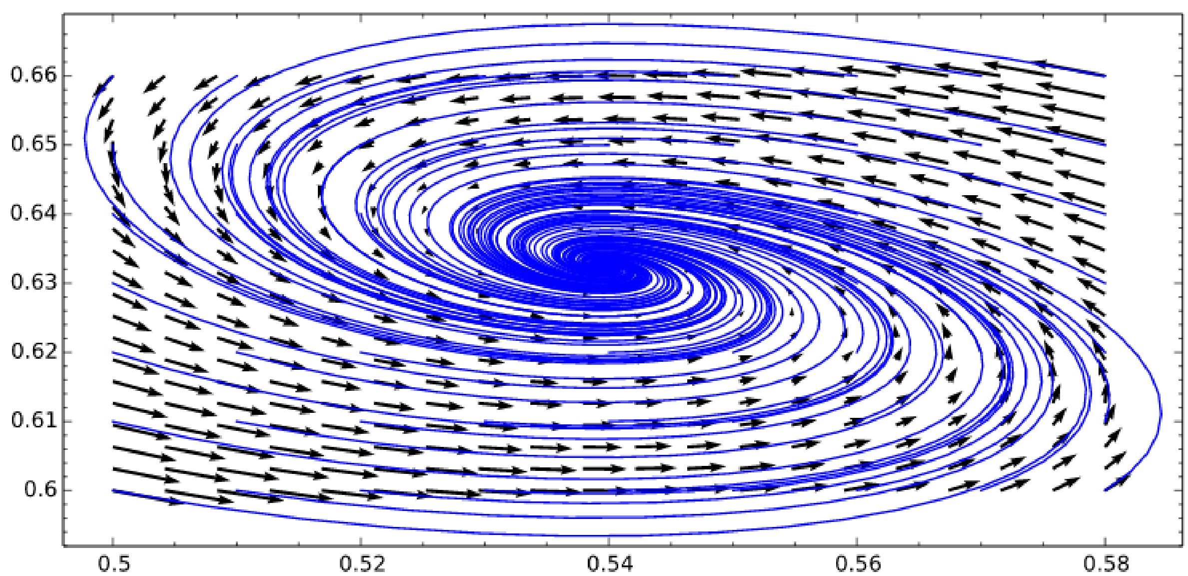
```
VF = plot_vector_field([f(x, y), g(x, y)], [x, 0.5278, 0.528], [y, .0042, 0.00435])
T = ode_solver()
T.function = lambda t, y: [f(y[0], y[1]), g(y[0], y[1])]

solutions = []
for x_0 in range(21112,21121):
    for y_0 in range(168,175):
        T.ode_solve(y_0 = [x_0/40000, y_0/40000], t_span = [0, 400], num_points = 160)
        solutions.append(line([p[1] for p in T.solution]))
```



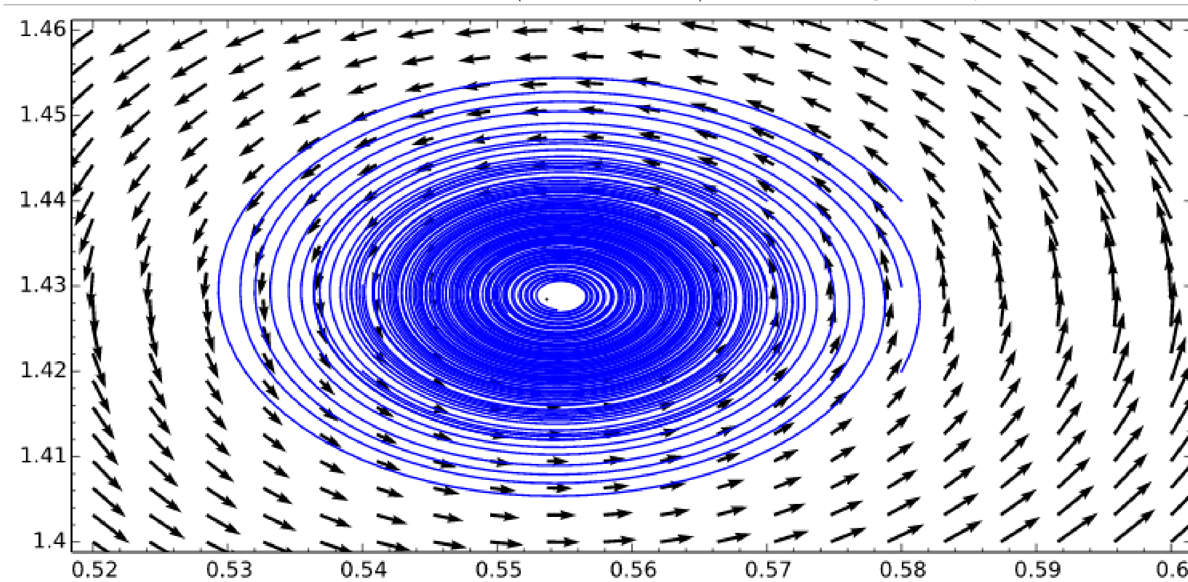
If $a = 0.4$ then equilibrium is a sink with (tr,det)$\approx (-0.36, 0.00047)$ The sink has det-trace values very close to the saddle region. Zooming way in for a phase portrait around this sink, as shown below, we still see very abrumpt transition from left/right motion to snapping to the eigenline and approaching the sink near $(0.53, 0.108)$, now somewhat more gently, still a sink:



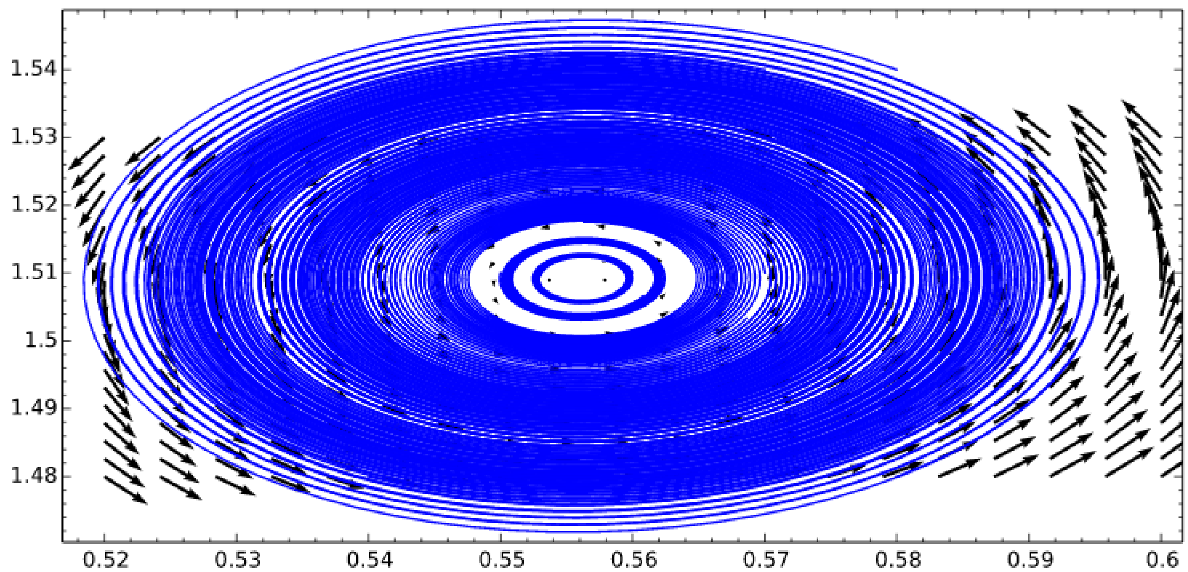
If $a = 0.8$ then equilibrium is solidly in the spiral-sink zone with (tr,det)$\approx (-0.19, 0.069)$ The phase portrait below is centered on the equilibrium at $(0.5396, 0.6325)$:



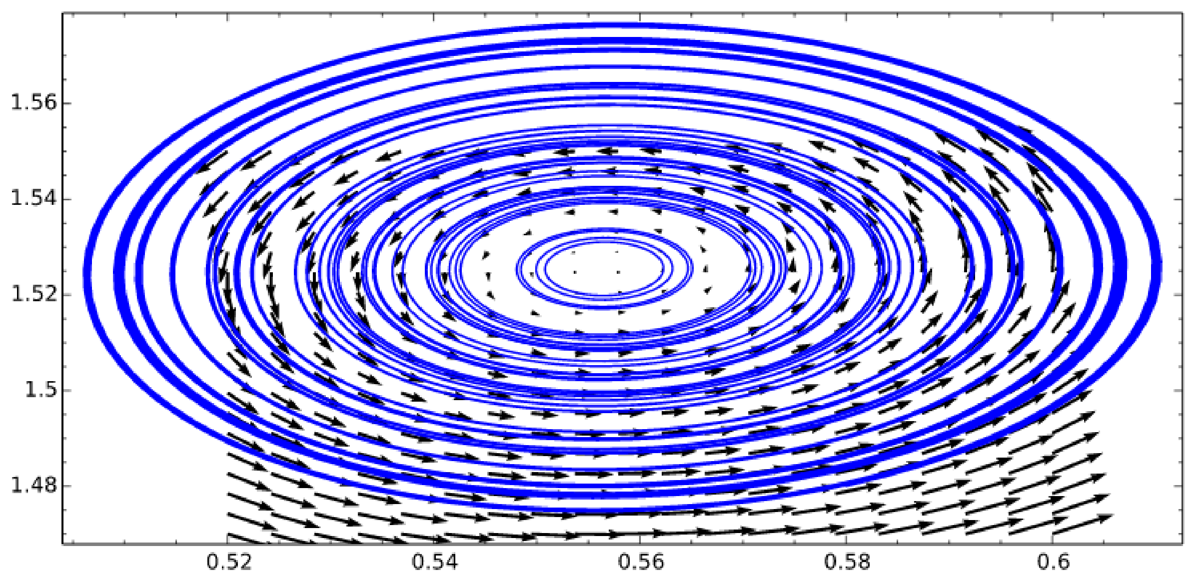
At $a = 1.4$ we are still in the spiral-sink zone, with (tr,det)$\approx (-0.021, 0.155)$ but we approaching the Hopf point. The orbits approach the equilibrium at $(0.5547, 1.4288)$ much more gradually:



At $a = 1.46$ we are just barely in the spiral-sink zone, with (tr,det)$\approx (-0.0035, 0.164)$. Here, the orbits approach the equilibrium at $(0.5562, 1.5092)$ even more gradually, seemingly like to birth a limit cycle:



At $a = 1.47214$ orbits are pretty much closed curves (no more spirals), with (tr,det)$\approx (1.1e-8, 0.1654)$. The equilibrium at $(0.5565, 1.5255)$ is the center of these concentric oval orbits:



At $a = 2$ orbits are clearly in spriral source mode, with (tr,det)$\approx (0.16, 0.24)$. The equilibrium at $(0.57103, 2.2384)$ is the source center: