1 Purpose

To examine ODE models of the interactions of species, to make predictions about changes in the populations, and to study sensitivity to changes in rate constants.

2 Background

2.1 Predator Prey

etc...let's just look at

2.2 Satiable Predation

When food is plentiful, the predator's appetite is satiated, and an increase in prey population has little effect on the interaction terms in the rate equations. One model for this is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -a & \dfrac{bx}{k+y} \\ \dfrac{-dy}{k+y} & c-\gamma y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

where prey overcrowding is also present.

Let's see how this could be modeled using Sage. First, I'll try to solve it analytically:

```
var('x y t')
x = function('x',t)
y = function('y',t)
de1 = (diff(x,t)==-x+x*y/(20+y));show(de1)
de2 = (diff(y,t)==-x*y/(20+y)+y*(1-y/50));show(de2)
sol = desolve_system([de1,de2],[x,y],[0,5,1],t);show(sol)
f(t)=sol;show(f)
```

This produces the dizzying response:

$$D[0]\left(x\right)\left(t\right) = \frac{x\left(t\right)y\left(t\right)}{y\left(t\right)+20} - x\left(t\right)$$

$$D[0]\left(y\right)\left(t\right) = -\frac{1}{50}\left(y\left(t\right)-50\right)y\left(t\right) - \frac{x\left(t\right)y\left(t\right)}{y\left(t\right)+20}$$

$$x\left(t\right) = \mathcal{L}^{-1}\left(\frac{\mathcal{L}\left(\frac{x(t)y(t)}{y(t)+20}, t, g_{2421}\right)+5}{g_{2421}+1}, g_{2421}, t\right)$$

$$y\left(t\right) = \mathcal{L}^{-1}\left(-\frac{\mathcal{L}\left(y\left(t\right)^{2}, t, g_{2421}\right)+50\,\mathcal{L}\left(\frac{x(t)y(t)}{y(t)+20}, t, g_{2421}\right)-50}{50\left(g_{2421}-1\right)}, g_{2421}, t\right)$$

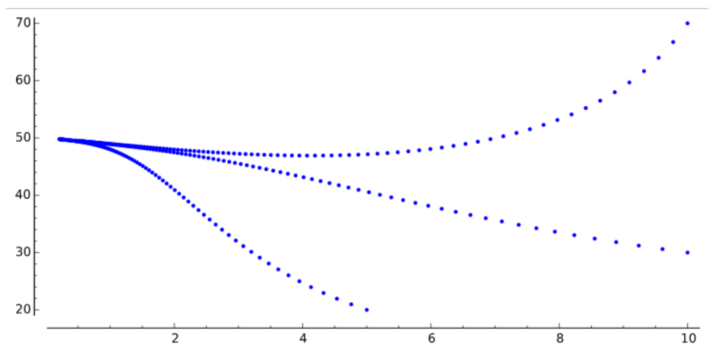
$$t \mapsto \left(x\left(t\right) = \mathcal{L}^{-1}\left(\frac{\mathcal{L}\left(\frac{x(t)y(t)}{y(t)+20}, t, g_{2421}\right)+5}{g_{2421}+1}, g_{2421}, t\right), y\left(t\right) = \mathcal{L}^{-1}\left(-\frac{\mathcal{L}\left(y\left(t\right)^{2}, t, g_{2421}\right)+50\,\mathcal{L}\left(\frac{x(t)y(t)}{y(t)+20}, t, g_{2421}\right)-50}{50\left(g_{2421}-1\right)}, g\right.$$

which I won't even try to fit on the page. What is that script L, the Laplacian maybe? Worse, what the heck is g_{2421}. It makes you wonder about the state of Sage...clearly, what we want is the numerical solver:

```
from sage.calculus.desolvers import desolve_system_rk4

x,y,t=var('x y t')
#x=function('x',t)
P=desolve_system_rk4([-x+x*y/(20+y),-x*y/(20+y)+y*(1-y/50)],[x,y],ics=[0,5,20],ivar=t) #,end_points=2
P=P+desolve_system_rk4([-x+x*y/(20+y),-x*y/(20+y)+y*(1-y/50)],[x,y],ics=[0,10,30],ivar=t) #,end_point
P=P+desolve_system_rk4([-x+x*y/(20+y),-x*y/(20+y)+y*(1-y/50)],[x,y],ics=[0,10,70],ivar=t) #,end_point
Q=[ [i,j] for i,j,k in P]
LP=list_plot(Q)
# Q=[ [j,k] for i,j,k in P]
Q=[ [j,k] for i,j,k in P]
LP=list_plot(Q)
show(LP)
```

This produces a nice plot at right which shows how the three initial conditions $(x_0, y_0) = (5, 20), (10, 30), (10, 70)$ and how they all lead to death of the predator and the prey at carrying capacity.

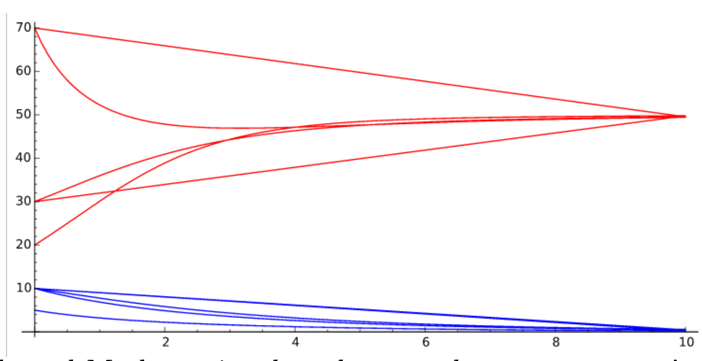


But how do you get the x vs. t and y vs. t plots? Now, I posed the question to the SageMath support group today (3/23/15) and received about an hour later the following response:

```
The command desolve_system_rk4 returns a list of lists of the form
[time pt, x value, y value]. You can use a list comprehension as below to
pull out coordinate pairs. Then, you can plot using the line command.

x,y,t=var('x y t')
P=desolve_system_rk4([x*(1-y),-y*(1-x)],[x,y],ics=[0,0.5,2],ivar=t,end_points=20)
tx = [[q[0],q[1]] for q in P]
ty =  [[q[0],q[2]] for q in P]
line(tx)+line(ty,color='red')
```

The red curves show the prey and the blue curves the predator. I'm not sure why there appears to be one more blue curve than red curve.



Ok, I'd like to give you some crib sheets on Maple and Mathematica, but they are the enemy, so you're on your own with them...

Instead, let's look at another free-ware package from Borrelli and Coleman of Harvey Mudd: ODE Toolkit. Look at the help file - it's fairly simple to use!

2.3 Mutualism

The system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & bx \\ dy & c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

describes a symbiotic interspecies relationship that is unbounded by the environment.

Observation These models make sense only in the **population quatdrant**, $x \geq 0, y \geq 0$ An orbit starting in that quadrant always stays there since the positive axes are unions of orbits; the Uniqueness Principle prevents any other orbit from touching the axial orbits.

Observation The models have their flaws. No account is taken of the time delay between an action and its effect on population rates. Averaging the rates of change over all categories of age, sex, fertility, and health is of dubious validity. More intricate models could be (and have been) constructed, but most of these have their origins in the models presented above.