**To solve a separable first-order differential equation $x'(t) = g(t)h(x)$:**

- Write the equation in the form $dx/dt = g(t)h(x)$.
- Multiply both sides by $dt$, divide by $h(x)$, and integrate, to put the equation in the form

$$\int [1/h(x)]dx = \int g(t)dt.$$

- Find any function $H(x)$ such that $H'(x) = 1/h(x)$ and any function $G(t)$ such that $G'(t) = g(t)$.
- Write the solution as $H(x) = G(t) + C$.
- If possible, solve the equation from the previous step explicitly for $x$, as a function of $t$.

The next example shows how this method works.

**Example 2.1.3.** Solve the separable differential equation $x' = t(x+1)$.

*Solution* First write the equation in the form

$$dx/dt = t(x+1). \tag{2.7}$$

Separate the variables (including $dx$ and $dt$) so only the variable $x$ appears on the left and $t$ on the right:

$$\frac{1}{(x+1)}dx = tdt.$$

Integration of each side with respect to its own variable of integration leads to the implicit solution

$$\ln|x+1| = t^2/2 + C,$$

which can be solved explicitly for $x$ by applying the exponential function:

$$e^{\ln|x+1|} = |x+1| = e^{(t^2/2+C)} = e^C e^{(t^2/2)}.$$

If the positive constant $e^C$ is replaced by a nonzero constant $A$ that can be either positive or negative, the absolute value signs can be dropped, to give

$$x + 1 = Ae^{(t^2/2)},$$

where $A = \pm e^C$. Then the explicit solution is

$$x(t) = Ae^{(t^2/2)} - 1. \tag{2.8}$$

■

The formula for $x(t)$ is the one-parameter family of curves that we expect to get as the solution of a first-order differential equation, and notice that the parameter $A$ was introduced in the step where the equation was integrated.

As we saw in Section 1.2, a **particular solution** to a first-order differential equation is a solution in which there are no arbitrary constants. It will be shown (in Section 2.4) that, in general, to obtain a particular solution of a first-order differential equation it is necessary and sufficient to give one initial condition of the form $x(t_0) = x_0$.

Again from Section 1.2, we call a one-parameter family of solutions of a first-order differential equation, containing a single constant of integration, a **general solution** if it contains every solution of the equation. The analytic solution of a separable equation, found by the method just described, will contain all solutions of the equation with the possible exception of **constant solutions.** These are solutions of the form $x(t) \equiv C$ which make the right-hand side of the differential equation $x' = f(t, x)$ identically equal to 0. Note that with $x \equiv C$ the differential equation is satisfied, because the derivative of a constant function is also zero; that is, $\frac{d}{dt}(C) \equiv 0 \equiv f(t, C)$.

Referring back to the example 2.1.3, the function $x \equiv -1$ is the only constant solution of the differential equation $x' = t(x + 1)$. In this case it is given by the solution formula (2.8) when the constant $A = 0$; therefore, (2.8) is the general solution of (2.7) if we allow the value $A = 0$.

When solving a separable equation it is wise to find all constant solutions first, since they may be lost when the equation is divided by $h(x)$.

**Example 2.1.4.** Solve the initial-value problem $x' = t/x,\ x(0) = 1$.

*Solution* First note that this differential equation has no constant solutions; that is, there are no constant values for $x$ that make $t/x \equiv 0$. Write the equation as $dx/dt = t/x$. Then by multiplying by $dt$ and $x$, and integrating,

$$xdx = tdt \Longrightarrow \int xdx = \int tdt \Longrightarrow x^2/2 = t^2/2 + C.$$

The expression $x^2/2 = t^2/2 + C$ is an implicit solution, and yields two explicit solutions

$$x(t) = \pm\sqrt{t^2 + 2C}.$$

We can satisfy the initial condition by substituting $t = 0$ into the general solution and setting $x(0) = 1$:

$$x(0) = \pm\sqrt{0 + 2C} = 1.$$

This implies that $C$ must be 1/2 and the sign of the square root must be taken to be positive. Now the unique solution to the initial-value problem, $x(t) = \sqrt{t^2 + 1}$, is completely determined. ■

The following two applications show how separable differential equations and initial-value problems can arise in real-world situations.

### 2.1.1 Application 1: Population Growth

One of the simplest differential equations arises in the study of the growth of biological populations. Consider a population with size $P(t)$ at time $t$. If it is assumed that the population has a constant birth rate $\alpha$ and constant death rate $\beta$ per unit of time, then an equation for the rate of growth of the population is

$$dP/dt = \alpha P(t) - \beta P(t) = (\alpha - \beta)P(t) = rP(t), \tag{2.9}$$

where $r$ is called the *net growth rate* of the population. This is a separable differential equation with general solution (Check it!):

$$P(t) = Ke^{rt},$$

where $K$ is the arbitrary constant of integration. The initial value is frequently given as the size of the population at time $t = 0$. Then $P(0) = Ke^{r0} = K$, and the particular solution of this initial-value problem is $P(t) = P(0)e^{rt}$. This means that, $t$ units of time after the initial time, the population will have grown exponentially (or decreased exponentially if $\beta > \alpha$). Populations do not grow exponentially forever, and biologists usually use more complicated equations of growth to take this into account.

One assumption that can be made is that as the population $P$ increases, its growth rate decreases, due to the effects of crowding, intra-species competition, etc. The simplest way to decrease the growth rate as $P$ increases is to assume that the growth rate is linear in $P$; that is, replace $r$ in (2.9) by $R = r - \gamma P(t)$. Then

$$dP/dt = (r - \gamma P(t))P(t) = rP(t)\left(1 - \frac{\gamma}{r}P(t)\right) = rP(t)(1 - P(t)/N),$$

where we have defined a new constant $N = r/\gamma$. The equation

$$dP/dt = rP(t)(1 - P(t)/N) \tag{2.10}$$

is called the **logistic growth equation.**

Notice that the rate of growth $dP/dt$ goes to 0 as $P(t) \to N$. This limiting value of the population, $N$, is called the **carrying capacity** of the ecosystem in which the population lives. The parameter $r$, which now gives the approximate rate of growth when the population is small, is called the **intrinsic growth rate** of $P$.

The logistic growth equation (2.10) is an autonomous differential equation and therefore is separable, but the expression $dP/[P(1 - P/N)]$ has to be integrated using partial fractions (or with the use of computer algebra). In either case, using our technique for separable equations, we have

$$\frac{dP}{[P(1 - P/N)]} = r\,dt \Longrightarrow \int \frac{1}{P(1 - P/N)}dP = \int r\,dt. \tag{2.11}$$

To compute the integral on the left, we can use partial fractions to write

$$\frac{1}{P(1 - P/N)} \equiv \frac{1}{P} - \frac{1}{P - N}.$$

You should check this last equality carefully (there is a review of partial fraction expansions in Section 2 of Chapter 6).

Integration of (2.11), using the partial fraction expression, now results in

$$\ln|P| - \ln|P - N| = rt + K.$$

To solve for $P(t)$, apply the exponential function to both sides and use the properties of the exponential and logarithmic functions to write

$$\begin{aligned}
e^{\ln|P|-\ln|P-N|} &= e^{rt+K} \Longrightarrow \\
\frac{P}{P-N} &= K_1 e^{rt}, \text{ where } K_1 = \pm e^K \Longrightarrow \\
P &= PK_1e^{rt} - NK_1e^{rt} \Longrightarrow \\
P - PK_1e^{rt} &= -NK_1e^{rt} \Longrightarrow \\
P &= \frac{-NK_1e^{rt}}{1-K_1e^{rt}} = \frac{NK_1e^{rt}}{K_1e^{rt}-1} = \frac{N}{1-Ce^{-rt}} \qquad (2.12)
\end{aligned}$$

where $C = \frac{1}{K_1}$. Note that the differential equation $P' = rP(1 - P/N)$ has two constant solutions $P \equiv 0$ and $P \equiv N$. Using the value $C = 0$ in the solution (2.12) gives $P = N$, but no finite value of $C$ makes this solution identically 0. To have a general solution, we must add the solution $P \equiv 0$ to the formula in (2.12).

In Section 2.4 the interval of existence of the solutions of the logistic population equation will be carefully examined and it will be shown that solutions with $P(0)$ between 0 and $N$ exist for all $t$. Solutions with $P(0) > N$ exist for all $t > 0$ but have a vertical asymptote at a negative value of $t$. Solutions with $P(0) < 0$ tend to $-\infty$ at a positive value of $t$, but these are not physically realizable as populations.

### 2.1.2 Application 2: Newton's Law of Cooling

Newton's Law of Cooling is a well-known law of physics that states that if a small body of temperature $T$ is placed in a room with constant air temperature $A$, the rate of change of the temperature $T$ is directly proportional to the temperature difference $A - T$. This law can be expressed in the form of a differential equation:

$$T'(t) = k(A - T(t)),$$

where $T(t)$ is the temperature of the small body at time $t$, $A$ is the surrounding (ambient) air temperature, and $k$ is a positive constant that depends on the physical properties of the small body. The only constant solution of the equation is $T(t) \equiv A$, which says that if the body is initially at the ambient temperature, it will remain there.

The equation can be seen to be separable, and can be solved by writing

$$dT/dt = k(A-T) \Longrightarrow \int \frac{dT}{A-T} = \int k\,dt \Longrightarrow -\ln|A-T| = kt + C \Longrightarrow$$
$$|A-T| = e^{-(kt+C)} \Longrightarrow A - T(t) = \alpha e^{-kt},$$

where $\alpha = \pm e^{-C}$ can be any positive or negative real number. The explicit solution is

$$T(t) = A - \alpha e^{-kt}. \qquad (2.13)$$

The constant solution $T \equiv A$ is obtained from the formula by letting $\alpha$ have the value zero. The long-term behavior is very easy to determine here, since if $k > 0$, then $T(t) \to A$ as $t \to \infty$. Thus the temperature of the small body tends to the constant room temperature, which makes good sense physically.

Consider the following very practical example that uses Newton's Law of Cooling.

**Example 2.1.5.** A cup of coffee, initially at temperature $T(0) = 210°$, is placed in a room in which the temperature is $70°$. If the temperature of the coffee after five minutes has dropped to $185°$, at what time will the coffee reach a nice drinkable temperature of $160°$?

*Solution* If we assume the cup of coffee cools according to Newton's Law of Cooling, the general solution given by (2.13), with $A = 70$, can be used to write

$$T(t) = 70 - \alpha e^{-kt}.$$

Using the initial condition, we can find the value of $\alpha$:

$$T(0) = 70 - \alpha e^{-0k} = 70 - \alpha = 210 \Longrightarrow \alpha = -140.$$

The temperature function can now be written as $T(t) = 70 + 140e^{-kt}$. To find the value of the parameter $k$, use the given value $T(5)$:

$$T(5) = 70 + 140e^{-5k} = 185 \Longrightarrow e^{-5k} = \frac{115}{140} \Longrightarrow$$

$$k = -\frac{1}{5}\ln(\frac{115}{140}) \approx 0.0393.$$

The value for $k$ completely determines the temperature function; that is,

$$T(t) = 70 + 140e^{-0.0393t}$$

for all $t > 0$. Now the answer to the original question can be found by solving the equation $T(\hat{t}) = 160$ for $\hat{t}$. The approximate value for $\hat{t}$ is 11.2 minutes. ■

In the last example, if the value of the physical parameter $k$ had been known beforehand, only one value of the temperature would have been required to determine the function $T(t)$ exactly. In this problem, the value of the parameter $k$ had to be determined experimentally from the given data, thus necessitating the temperature to be read at two different times. This sort of thing is even more likely to occur in problems that come from non-physical sciences, where parameters are usually not known physical constants and must be experimentally determined from the data provided.

**Exercises 2.1.** In problems 1–4, determine whether the equation is separable.

1. $x' + 2x = e^{-t}$
2. $x' + 2x = 1$
3. $x' = \frac{x+1}{t+1}$
4. $x' = \frac{\sin t}{\cos x}$

Put equations 5–14 into the form $x'(t) = g(t)h(x)$, and solve by the method of separation of variables.

5. $x' = \frac{x}{t}$
6. $x' = \frac{t}{x}$
7. $x' = x + 5$

8. $x' = 3x - 2$
9. $x' = x\cos(t)$
10. $x' = (1 + t)(2 + x)$
11. $xx' = 1 + 2t + 3t^2$
12. $x' = (t + 1)(\cos(x))^2$
13. $x' = t + tx^2$
14. $x' = 2 - tx^2 - t + 2x^2$ (Hint: factor.)

In 15–20, solve the initial-value problem.

15. $y' = y + 1, y(0) = 2$
16. $y' = ty, y(0) = 3$
17. $x' = x\cos(t), \quad x(0) = 1$
18. $x' = (1 + t)(2 + x), \quad x(0) = -1$
19. $x' = (t + 1)(\cos(x))^2, \quad x(0) = 1$
20. $P' = 2P(1 - P), \quad P(0) = 1/2$
21. (Newton's Law of Cooling) A cold Pepsi is taken out of a 40° refrigerator and placed on a picnic table. Five minutes later the Pepsi has warmed up to 50°. If the outside temperature remains constant at 90°, what will be the temperature of the Pepsi after it has been on the table for twenty minutes? What happens to the temperature of the Pepsi over the long term?
22. (Newton's Law of Cooling) Disclaimer: The following problem is known not to be a very good physical example of Newton's Law of Cooling, since the thermal conductivity of a corpse is hard to measure; in spite of this, body temperature *is* often used to estimate time of death.

    At 7 AM one morning detectives find a murder victim in a closed meat locker. The temperature of the victim measures 88°. Assume the meat locker is always kept at 40°, and at the time of death the victim's temperature was 98.6°. When the body is finally removed at 8 AM, its temperature is 86°.

    (a) When did the murder occur?

    (b) How big an error in the time of death would result if the live body temperature was known only to be between 98.2° and 101.4°?
23. (Orthogonal Curves) In the exercises for Section 1.3 the family of curves orthogonal to the family $\bar{y} = cx^3$ was shown to satisfy the differential equation $y' = -\frac{x}{3y}$. Use the method for separable equations to solve this equation, and find a formula for the orthogonal family.
24. (Orthogonal Curves) Show that the family of curves orthogonal to the family $\bar{y} = ce^x$ satisfies the equation $y' = -\frac{1}{y}$ (you may want to refer back to the model labeled MATHEMATICS in Section 1.3). Solve this separable equation and plot three curves from both families. If you make the scales identical on both axes, the curves should appear perpendicular at their points of intersection.

25. (Population Growth)

In this problem you are asked to compare two different ways of modeling a population; either by a simple exponential growth equation, or by a logistic growth model.

**Census Table**

| $t$ | *year* | *pop(millions)* |
|---|---|---|
| 0 | 1800 | 5.2 |
| 0.5 | 1850 | 23.2 |
| 1.0 | 1900 | 76.2 |
| 1.5 | 1950 | 151.3 |
| 2.0 | 2000 | 281.4 |

One population for which reasonable data is available is the population of the United States. The Census Table above gives census data for the population (in millions) from 1800 to 2000 in 50 year intervals. If the logistic function $P(t) = \frac{N}{1+Ce^{-rt}}$ is fit exactly to the three data points for 1800, 1900, and 2000, the values for the parameters are $N = 331.82$, $C = 62.811$, and $r = 2.93$, and the equation becomes

$$P(t) = \frac{331.82}{1 + 62.811e^{-2.93t}}.$$

Note that $t$ is measured in hundreds of years, with $t = 0$ denoting the year 1800.

(a) Plot (preferably using a computer) an accurate graph of $P(t)$ and mark the five data points on the graph (three of them should lie *exactly* on the curve).

(b) Where does the point $(t, P(t)) = (1.5, 151.3)$ lie, relative to the curve? Can you think of a reason why — famine, disease, war?

(c) What does the logistic model predict for $P(2.1)$; that is, the population in 2010? (The census data gives 308.7 million.)

(d) What does the model predict for the population in 2100? Does this seem reasonable?

(e) Now fit a simple exponential model $p(t) = ce^{rt}$ to the same data, using the two points for the years 1900 and 1950 to evaluate the parameters $r$ and $c$. What does this model predict for the population in 2100? Does this seem more or less reasonable than the result in (d)?
Note: If you use $t = 0$ for 1900 and $t = 1$ for 1950, then the population in 2100 is $p(4)$.

**COMPUTER PROBLEMS:** Use your computer algebra system to solve the equation in each of the odd-numbered exercises 5–13 above. The Maple or *Mathematica* instructions you need are given at the end of the exercises in Section 1.2. You can use the answers in Appendix A to check your computer results.

## 2.2 Graphical Methods, the Slope Field

For any first-order differential equation

$$x' = f(t, x), \qquad (2.14)$$

whether or not it can be solved by some analytic method, it is possible to obtain a large amount of graphical information about the general behavior of the solution curves from the differential equation itself. In Section 2.4 you will see that if the function $f(t,x)$ is everywhere continuous in both variables $t$ and $x$ and has a continuous derivative with respect to $x$, the family of solutions of (2.14) forms a set of non-intersecting curves that fill the entire $(t,x)$-plane. In this section we will see how to use the slope function $f(t,x)$ to sketch a field of tangent vectors (called a **slope field**) that show graphically how the solution curves (also called **trajectories**) flow through the plane. This can all be done even when it is impossible to find an exact formula for $x(t)$.

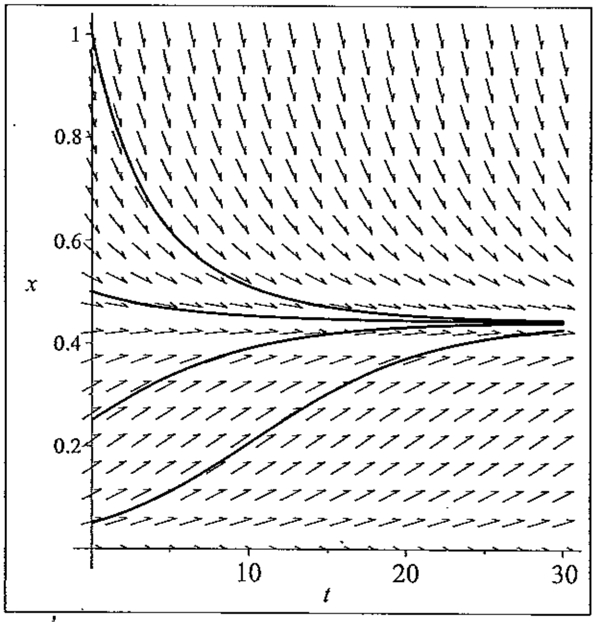


**Figure 2.1.** Slope field for the equation $x' = 0.2x(1-x) - 0.3x^2/(1+x^2)$

Figure 2.1 shows a slope field for the spruce-budworm equation, which has the general form

$$x' = rx\left(1 - \frac{x}{N}\right) - \frac{ax^2}{b^2 + x^2}.$$

This equation models the growth of a population of pests that attack fir trees. It is essentially a logistic growth equation with an added term that models the effect of predation on the pests, primarily by birds.

You should try to solve this equation with your CAS. It may print out some long unintelligible formula, but basically it is not able to solve the equation in terms of elementary functions, such as polynomials, exponentials, sines and cosines, etc. Notice, however, how easy it is to see how the family of solutions behaves. In Figure 2.1, Maple was used to draw the slope field and four numerically computed solutions (much more will be said about numerical solutions in Section 2.6).

To understand how a slope field is drawn, let $x(t)$ be a solution of (2.14). Then if $x(t)$ passes through some point $(\bar{t}, \bar{x})$ in the plane, the differential equation states that the graph