$x(0) = x_0$ we can find the corresponding value of C. But there is a definite difference between this family of curves and the family shown in Figure 1.1. Can you spot it?

Every solution except the zero solution becomes infinite at a vertical asymptote. For example, to find the solution satisfying the initial condition $x(1) = 2$ we would set

$$x(1) = \frac{1}{C-1} = 2$$

and solve for C. Thus $C - 1 = \frac{1}{2}$ and $C = \frac{3}{2}$. The particular solution through this initial point is

$$x(t) = \frac{1}{\frac{3}{2} - t}$$

and it can be seen that $\lim_{t \to \frac{3}{2}^-} \left(\frac{1}{\frac{3}{2} - t} \right) = +\infty$. This curve, with its vertical asymptote, is shown in Figure 1.2.

This example is meant to make you suspicious of what can happen when you solve nonlinear equations. There are nice theorems for linear equations that give you exact intervals in which solutions are guaranteed to exist. The comparable theorems for nonlinear equations usually do not guarantee existence of a solution except in some small unspecified interval sufficiently close to the given initial point.

Exercises 1.2. In problems 1–4, use differentiation to show that the given function is a solution of the equation for all values of the constants.

1. Equation: $x' = x - 2$, function $x = 2 + Ce^t$.
2. Equation: $\frac{dy}{dx} = 4(x+1)y$, function $y = Ce^{2x^2+4x}$.
3. Equation: $\frac{dy}{dt} = y + t$, function $y = Ce^t - 1 - t$.
4. Equation: $x'' + x = 2e^t$, function $x = C_1 \sin(t) + C_2 \cos(t) + e^t$.

Solve the initial value problems 5–10, using the given family of solutions. In each case, first show that the family of solutions satisfies the equation for all values of the constants. Also state the exact interval in which the particular solution of the initial-value problem exists.

5. $x' = 2x$, $x(0) = -1$; family of solutions $x = Ce^{2t}$.
6. $y' = y^2$, $y(0) = -\frac{1}{2}$; family of solutions $y = \frac{1}{C-t}$.
7. $y' = 1 + y^2$, $y(0) = 1$; family of solutions $y = \tan(t + C)$.
8. $\frac{dx}{dt} = 2tx$, $x(0) = 1$; family of solutions $x = Ce^{t^2}$.
9. $x'' + x = t^2$, $x(0) = 0$, $x'(0) = 1$; family of solutions $x(t) = C_1 \sin(t) + C_2 \cos(t) + t^2 - 2$.
10. $x'' + 2x' + x = 0$, $x(0) = 1$, $x'(0) = -1$; family of solutions $x = C_1 e^{-t} + C_2 t e^{-t}$.
11. For the differential equation $x' = x^3$,
    (a) Show that $x(t) = \sqrt{\frac{1}{C-2t}}$ is a one-parameter family of solutions.
    (b) Find one solution of $x' = x^3$ that cannot be written in this form for any value of C.

(c) Find the solution satisfying the initial condition $x(0) = 1$, and sketch its graph.

(d) Does the solution in part (c) have a vertical asymptote? If so, for what value of t?

COMPUTER PROBLEMS: In Maple, the instructions

`dsolve(ODE)` or `dsolve({ODE,initcond},y(t))`

can be used to produce the solution of an ordinary differential equation (ODE), if an analytic solution exists. If no initial conditions are specified, the solution will contain one or more parameters. The corresponding instructions in *Mathematica* are

`DSolve[ODE,y[t],t]` or `DSolve[{ODE,initcond},y[t],t]`.

12. Use a computer to solve the differential equation

$$\frac{dy}{dt} + 2y(t) = 1. \tag{1.9}$$

The computer instructions are

`dsolve(y'(t)+2*y(t)=1)` in Maple

`DSolve[y'[t]+2 y[t]==1,y[t],t]` in *Mathematica*.

In Maple the multiplication operator $*$ cannot be omitted. Write the computer solution and check that it satisfies (1.9). How is the parameter represented in the computer solution?

13. Either by hand, or by executing the computer instruction

Maple: `dsolve({y'(t)+2*y(t)=1,y(0)=y0},y(t))`

Mathematica: `DSolve[{y'[t]+2y[t]==1,y(0)==y0},y[t],t]`

five times, with five different values for $y(0)$, solve equation (1.9) with initial conditions $y(0) = -1, 0, 1, 2,$ and 3. Use an ordinary plotting routine to plot the five solutions on the same set of axes, over the interval $-5 \leq t \leq 5$. Describe, as precisely as you can, the behavior of this set of curves. Especially note the behavior of $y(t)$ as $t \to \infty$.

14. Use your own computer algebra system to solve the equation $x'' + x = t^2$. The equation should be entered in the form $x''(t) + x(t) = t^2$. Do you get the solution given in Exercise 9? How are the two parameters represented in the computer solution?

1.3 Modeling with Differential Equations

To give you an idea of how differential equations arise in real-world problems, and what methods are used to solve them, this section presents examples from five different fields of science. By the time you have worked through Chapter 2 you will have a method for solving all of the first-order equations that appear in these five examples.

Some can be solved analytically by finding an algebraic formula for the solution, but others cannot. In the cases where no analytic solution exists, Maple has been used to draw both a slope field and some numerically computed solution curves. The slope field consists of a collection of short arrows that are drawn tangent to the solution curves. Much more will be said about slope fields in Section 2.2 of Chapter 2.

I. PHYSICS

The first problem comes from physics, where we want to model the velocity $v(t)$ of a skydiver falling from a plane. In free fall, Newton's second law can be used to write mass $\times$ acceleration = sum of forces. Since acceleration is the derivative of velocity, that is, $a(t) = v'(t)$, this results in the differential equation

$$mv'(t) = mg - k(v(t))^p,$$

where v is the velocity in meters/second, m is the mass in kilograms of the falling skydiver, $g = 9.8$ m/s^2 is the downward acceleration due to gravity, k is the coefficient of friction due to air resistance, and p is an exponent usually assumed to be equal to one. In the case $p = 1$, which implies that the friction due to the air resistance is directly proportional to the velocity, the differential equation can be solved analytically, and an exact formula

$$v(t) = \frac{mg}{k} + Ce^{-\frac{k}{m}t}$$

can be found for the velocity at time t. This is a one-parameter family of solutions. It contains a single constant C, which can be determined by specifying a single initial condition of the form $v(t_0) = v_0$. If $p \neq 1$, the problem becomes nonlinear and therefore harder to solve analytically. In this case a numerical solution can be found for any initial value. In Figure 1.3 numerically obtained solutions corresponding to an arbitrarily chosen set of initial values are shown for the equation with parameters $m = 72$, $g = 9.8$, $p = 0.9$, and $k/m = 0.6$. It is clear from the figure that the velocity tends to a constant value as $t \to \infty$; this is called the terminal velocity. When $p = 1$ the exact formula for $v(t)$ shows that the terminal velocity, which is equivalent to $\lim_{t\to\infty} v(t)$, is $\frac{mg}{k}$.

When we study autonomous equations in Section 2.7 we will see that for any positive p, the terminal velocity is

$$\lim_{t\to\infty} v(t) = \left(\frac{mg}{k}\right)^{1/p}.$$

Does this value agree with the terminal velocity pictured in Figure 1.3?

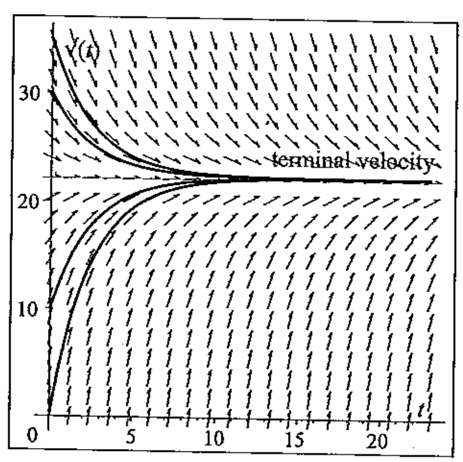


Figure 1.3. Solutions of $v' = 9.8 - 0.6v^{0.9}$

II. MATHEMATICS

A problem that arises in several areas of applied mathematics is that of finding a one-parameter family of curves $y = F(x, C)$ that is **orthogonal** to a given family $\bar{y} = f(x, c)$ at each point in the (x, y)-plane. To make the two families of curves orthogonal, at each point in the plane their slopes must be negative reciprocals of each other. This means that at each point (x, y), if the slope of $\bar{y}(x)$ is m, the slope of the curve $y(x)$ must be $-1/m$.

To see how this results in a differential equation, let the given set of curves be the one-parameter family of parabolas $\bar{y} = cx^2$. The slope of a curve in this family is $m = \bar{y}' = \frac{d}{dx}(cx^2) = 2cx$, and using the equation for $\bar{y}$ we have $c = \bar{y}/x^2$, so $m = 2(\bar{y}/x^2)x = 2(\bar{y})/x$. This means that the curve in the orthogonal family will have slope $-x/2y$ at the point (x, y); hence the functions $y(x, C)$ in the orthogonal family must satisfy the differential equation $y' = -x/2y$.

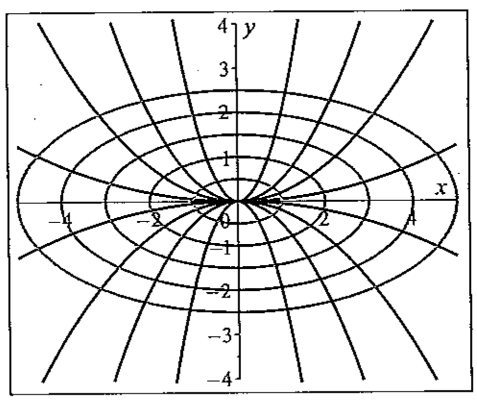


Figure 1.4. Orthogonal Curves

You will be shown in Chapter 2 that this is a "separable" differential equation, and you will be given a method for obtaining a formula for the solution. For the equation $y' = -x/2y$, the solution can be written as an implicit function of the form $y^2 + x^2/2 = C$, which can be seen to be a one-parameter family of ellipses. Figure 1.4 shows a set of solution curves from each of the two families. These were simply plotted with a plot routine by using the formulas $\bar{y} = cx^2$ and $y^2 + x^2/2 = C$ with arbitrarily chosen values of the constants c and C.

III. ENGINEERING

The current $i(t)$ in amperes in a simple electrical circuit containing a resistor, an inductor, and a sinusoidal electromotive source can be modeled by the equation $Li'(t) + Ri(t) = p\sin(t)$, where L is the inductance in henrys, R is the resistance in ohms, and $E(t) = p\sin(t)$ is a periodic electromotive force in volts. This equation turns out to be similar to a simplified version of the spring-mass equation (studied in Chapter 3), and is linear in the dependent variable i. Because the equation is linear with constant coefficients L and R, it is possible to find a formula for the general solution by an analytic method. For this equation the solution will be found to be

$$i(t) = Ce^{-\frac{R}{L}t} + \frac{p}{L^2 + R^2}(R\sin(t) - L\cos(t)).$$

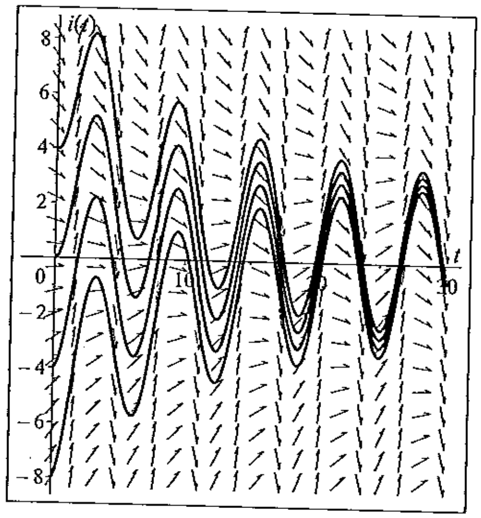


Figure 1.5. Solutions of $i'(t) + 0.1i(t) = 3\sin(t)$

When we study the uniqueness and existence of solutions in Section 2.4 we will see that this one-parameter family of curves fills the plane, and no two curves in the family can intersect.

For the equation with $L = 1$, $R = 0.1$, and $p = 3$, four solution curves are shown in Figure 1.5, each satisfying a different initial condition at $t = 0$. These curves were drawn by a Maple program that solves differential equations and plots a slope field of tangent vectors to the solution curves. This means that even though an analytic formula exists for the solution, so that you could plot the curves with a simple plot routine, it is possible to check the solutions using the differential equations program. It can be seen, both from the formula for the solution and from the graph itself, that in the limit as $t \to \infty$ the curves $i(t)$ tend to the sinusoidal curve $\frac{p}{L^2+R^2}(R\sin(t) - L\cos(t))$. With the parameter values used for Figure 1.5, this steady state curve is

$$i(t) = \frac{3}{1.01}\left(0.1\sin(t) - \cos(t)\right).$$

The function $Ce^{-\frac{R}{L}t}$ in the formula for the solution is the transient, which dies out after the switch is closed. Much more will be said about this in Chapter 3.

IV. ECOLOGY

The use of differential equations in biology is currently increasing at a rapid pace. The particular equation $P'(t) = rP(t)(1 - P(t)/N)$, called the logistic growth equation, has been around for a long time, but new ways to use it are still being found. The particular version of the equation given here models the growth of a population of size $P(t)$, which is assumed to grow exponentially like e^{rt} when the size of the population is small. The parameter r is called the **intrinsic growth rate** of the population. As the population increases it approaches a limiting value N, called the **carrying capacity** of the population in the ecosystem in which it lives.

This simple version of the equation can be solved analytically, and the formula for the solution is $P(t) = N/(1 - Ce^{-rt})$. The graph in Figure 1.6 shows the characteristic

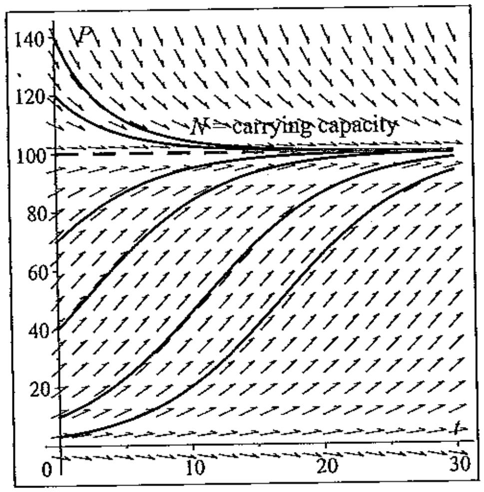


Figure 1.6. Solutions of the growth equation $P'(t) = 0.2P(t)(1 - P(t)/100)$

solution curves, all of which tend toward the carrying capacity if the initial condition is positive. For this application, negative initial conditions are not biologically meaningful.

The logistic growth equation has been used by ecologists to yield important information; for example, in the case of fish populations, it has been used to predict when they may be in danger of being depleted. In the remainder of the book, we will have much more to say about this equation and many other population models that are directly related to it.

V. NEUROLOGY

Our final example is another application of mathematics to biology, and is currently of interest to research mathematicians as well as to computational neuroscientists..The equation $x' = -x + S(x - \theta + e(t))$, is a simplistic version of an equation representing the activity level of certain nerve cells in the brain. The function $x(t)$ in this equation is assumed to be the percent of nerve cells that are active at time t in a certain region of the brain. The function $e(t)$ represents activity being received from cells outside the region, and θ is a threshold level common to all cells in the region. The function S is called a **response function**, and has shape similar to that of the function $P(t)$ in the previous example (see Figure 1.6). It is used to simulate the effect that a given amount of input to the cells has on the rate of growth of their activity level. The function S is often chosen to be of the form $S(z) = 1/(1 + e^{-Kz})$, which varies from 0 to 1 as z goes from $-\infty$ to $+\infty$. This is a highly nonlinear function, making the differential equation difficult to analyze. In fact no exact formula for the solution exists.

Figure 1.7 shows a graphical solution of the differential equation produced by Maple, where the solution curves have been obtained approximately by numerical methods. The input function is $e(t) = -0.3\cos(2\pi t)$, the threshold $\theta = 0.5$, and K is arbitrarily chosen to be 15. With periodic input $e(t)$ to the cells, some very interesting things can happen. From the graph it appears that there is at least one oscillatory solution around the 50% activity level, which means that with an appropriate initial condition the activity level will oscillate with approximately half of the cells active and half inactive at any time $t > 0$. If the initial condition is too small, the activity level will die out and if it is too large it will

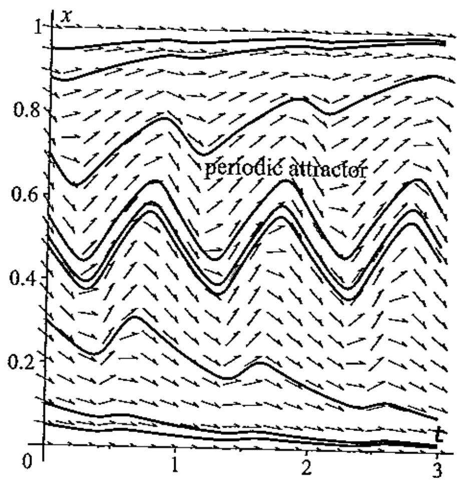


Figure 1.7. Solutions of the equation $x'(t) = -x(t) + 1/(1 + e^{-15(x(t)-0.5-0.3\cos(2\pi t))})$

approach a state where nearly 100% of the cells will be active. In a recent paper (R. Decker and V. W. Noonburg, A periodically forced Wilson-Cowan system with multiple attractors, *SIAM J. Math. Anal.* 2012, Vol. 44, pp. 887–905) a mathematical proof was given to show that there can actually be more than one periodic solution between the very high and very low levels of activity. This means that different initial states might cause different types of oscillatory behavior.

The five examples in this section are meant to show you how widespread the use of differential equations is. In the remainder of the book you will be given methods to solve these problems, and a great deal more. You will also be prepared for a more advanced course in dynamical systems. This is where you will learn what you need to know in order to read the current (and future) scientific literature, both in mathematics and even more importantly in applied fields such as engineering, biology, chemistry, ecology, and any of the many fields in which researchers are beginning to use advanced methods of mathematical analysis.

Exercises 1.3. To do the exercises for this section you will need to refer back to the given application.

- **PHYSICS**

  Assuming $p = 1$ and $g = 9.8m/s^2$, the equation $v' = g - \frac{k}{m}v$ has solution

  $$v(t) = \frac{mg}{k} + Ce^{-\frac{k}{m}t}.$$

  A man drops from a high flying plane and falls for 5 seconds before opening his parachute. With the parachute closed, $\frac{k}{m} = 1sec^{-1}$.

  (a) Find the man's velocity when he opens his parachute (use the initial condition $v(0) = 0$ to find the value of the constant C).

  (b) After the chute opens, what must the value of $\frac{k}{m}$ be to get his terminal velocity down to 5 mph? (Use 1 m/sec $\approx$ 2.237 mph.) Assume he has a very long way to fall.

- **MATHEMATICS**

Let $y(x) = cx^3$ be a one-parameter family of cubic polynomials.

(a) Show that the family of curves that is orthogonal at every point of the (x, y)-plane to this family of cubics satisfies the differential equation

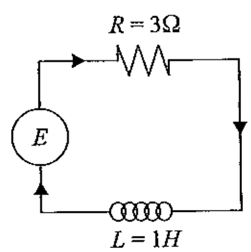
$$y' = -\frac{x}{3y}.$$

(b) Show that the set of curves $x^2 + 3y^2 = C$ is a one-parameter family satisfying the differential equation $y' = -\frac{x}{3y}$ in part (a).

(c) Either using a curve-plotting program, or by hand, sketch the curves $y = cx^3$ for $c = \pm 1$, $c = \pm 2$ and the curves $x^2 + 3y^2 = C$ for $C = 1, 2, 3$. If you make the scales on the x and y axes exactly the same, the curves should intersect at right angles.

- **ENGINEERING**

The diagram below shows a circuit containing the elements mentioned in the application; that is, a resistor with $R = 3$ ohms, an inductor with inductance $L = 1$ henry, and a generator producing a periodic electromotive force of $E = 10\sin(t)$ volts.



(a) The current $i(t)$ will approach a periodic steady state value in the limit as $t \to \infty$. Find the equation for the steady state curve.

(b) What initial value $i(0)$ could you use to produce this curve as the solution to the IVP for $Li'(t) + Ri(t) = E(t)$?

- **ECOLOGY**

A population of ants, initially containing 20 ants, is growing according to the population growth equation

$$P'(t) = 0.5P(t)\left(1 - \frac{P(t)}{N}\right),$$

with time t measured in days.

(a) If the carrying capacity of the area in which they live is 1000, how many ants will there be one week later?

(b) If bug spray is used, and it decreases the intrinsic growth rate r from 0.5 to 0.1, how many ants will you have one week later? Assume the carrying capacity remains unchanged at 1000 ants.

- **NEUROLOGY**

  The response function $S_K(z) = \frac{1}{1+e^{-Kz}}$ is used to simulate the effect of input on the rate of growth of the activity level of a cell.

  (a) On the interval $-1 \leq z \leq 1$, sketch graphs of $S_K(z)$ for $K = 4, 10,$ and 40. Which curve has the fastest rate of growth around $z = 0$? Notice that in every case, S is bounded between 0 and 1.

STUDENT PROJECT: In equations used to model activity of cells in the brain, other response functions $S(z)$ have been used to try to make the resulting differential equation easier to analyze. The main requirement is that a response function must increase monotonically from 0 to 1 as its argument goes from $-\infty$ to ∞.

(i) Find constants a and b that make

$$\tilde{S}_K(z) = \frac{\arctan(Kz) + a}{b}$$

a response function. Show graphically the effect of the parameter K.

(ii) Find constants c and d that make

$$\bar{S}_K(z) = \frac{\tanh(Kz) + c}{d}$$

a response function. Show graphically the effect of the parameter K.

(iii) Find constants K_1 and K_2 such that $\tilde{S}_{K_1}(z)$ and $\bar{S}_{K_2}(z)$ are close to the function $S_{15}(z) = \frac{1}{1+e^{-15z}}$. Do this graphically and plot the three functions on the same set of axes. Note: there is no *exact* answer to this problem.

(iv) Can you think of any other function that could be used as a response function? Remember that one goal is to make the function S as simple as possible.

CHAPTER 2

First-order

In this chapter, metho
ber that first-order m
derivative appearing i
differential equation l
will assume that the e
differential equation c

where f denotes an
makes sense, suppose

It would be messy, bu
tial equations of the f
also imagine equatio
some of our methods
The material in th
differential equations,
different graphical m
more specific method
on x. Numerical meth
existence and uniquen
with some real-world
of first-order equation

2.1 Separabl

The first analytic met
ten in the form

that is, when the func
function of x. Such a

CHAPTER 2

First-order Differential Equations

In this chapter, methods will be given for solving first-order differential equations. Remember that first-order means that the first derivative of the unknown function is the highest derivative appearing in the equation. While this implies that the most general first-order differential equation has the form $F(t, x, x') = 0$ for some function F, in this chapter we will assume that the equation can be solved explicitly for x'. This means that our first-order differential equation can always be put in the form:

$$x' = f(t, x), \tag{2.1}$$

where f denotes an arbitrary function of two variables. To see why such an assumption makes sense, suppose the differential equation is

$$(x'(t))^2 + 4x'(t) + 3x(t) = t.$$

It would be messy, but not impossible, to use the quadratic formula to extract two differential equations of the form $x' = f(t, x)$ from this quadratic equation. However, one could also imagine equations where solving for $x'(t)$ is not even possible, and in such a case, some of our methods may not be applicable.

The material in this chapter will cover several analytic methods for solving first-order differential equations, each requiring the function f in (2.1) to have a special form. Two different graphical methods are also described; one for the general equation (2.1), and a more specific method for an autonomous equation where f is a function depending only on x. Numerical methods for first-order equations are introduced, and theoretical issues of existence and uniqueness of solutions are discussed. In the examples you will be presented with some real-world problems in applied mathematics. Acquiring a solid understanding of first-order equations will lay the groundwork for everything that follows.

2.1 Separable First-order Equations

The first analytic method we will consider applies to first-order equations that can be written in the form

$$dx/dt = g(t)h(x); \tag{2.2}$$

that is, when the function $f(t, x)$ can be factored into a product of a function of t times a function of x. Such a differential equation is called **separable**.

Example 2.1.1. Determine which of the following first-order differential equations are separable. Hint: try to factor the right-hand side if the equation does not initially appear to be separable.

(a) $x' = xt + 2x$

(b) $x' = x + \cos(t)$

(c) $x' = xt^2 + t^2 - tx - t$

(d) $x' = x^2 + x + 3$

Solution Equation (a) can be factored into $x' = x(t+2)$, so it is separable with $g(t) = t + 2$ and $h(x) = x$. Equation (b) is not separable as $x + \cos(t)$ cannot be factored into a function of t multiplied by a function of x. With a bit of work, equation (c) can be factored as $x' = (t^2 - t)(x+1)$, so it is separable with $g(t) = (t^2 - t)$ and $h(x) = (x+1)$. Equation (d) can be written as $x' = (1)(x^2 + x + 3)$, with $g(t) = 1$ and $h(x) = x^2 + x + 3$, so it is separable. ■

Special Cases of $x' = g(t)h(x)$:

(1) If $h(x) = 1$, the separable equation $x' = g(t)$ is just an integration problem, and the solution is

$$x = \int g(t)dt;$$

that is, x is just the **indefinite integral** of the function $g(t)$. Remember that this means that x can be *any* function $G(t)$ such that $G'(t) = g(t)$, and this introduces an arbitrary constant into the solution. As an example, the solution of $x' = t + 1$ is

$$x(t) = \int (t+1)dt = \frac{t^2}{2} + t + C.$$

Even in this simple case the solution is an infinite one-parameter family of functions.

(2) If $g(t) = 1$, the separable equation $x' = h(x)$ is called an **autonomous** first-order differential equation. In this case it is no longer possible to solve the equation by simple integration, and the method given below must be used. Autonomous first-order differential equations are important and will be investigated more thoroughly in Section 2.7. In the above examples, only equation (d) is autonomous. The other three contain functions of t (other than the unknown function $x(t)$) on the right-hand side.

You may have already been shown, in calculus, an easy method for solving separable equations. Consider the following example.

Example 2.1.2. Solve the differential equation $\frac{dx}{dt} = -tx^2$.

Solution Split dx/dt into two pieces dx and dt, and do a bit of algebra to write

$$-\frac{dx}{x^2} = tdt.$$

Integrate each side with respect to its own variable to obtain

$$\int \left(-\frac{1}{x^2}\right) dx = \int tdt \Longrightarrow \frac{1}{x} = \frac{t^2}{2} + C, \tag{2.3}$$

where the arbitrary constants on each side have been collected on the right. Solve this equation for x to obtain the one-parameter family of solutions

$$x(t) = \frac{1}{t^2/2 + C}. \qquad \blacksquare$$

You should check that the function $x(t)$ does satisfy the differential equation for any value of the constant C. It appears that this method works, but splitting $\frac{dx}{dt}$ into two pieces is not a mathematically condoned operation; therefore, a justification of the method needs to be given.

If an equation is separable, and $x'(t)$ is written as dx/dt, both sides of the equation $dx/dt = g(t)h(x)$ can be divided by $h(x)$, and the equation becomes

$$\frac{1}{h(x(t))}\frac{dx}{dt} = g(t). \tag{2.4}$$

The two sides of (2.4) will be identical if, and only if, their indefinite integrals are the same up to an additive constant; that is,

$$\int \frac{1}{h(x(t))}\left(\frac{dx}{dt}\right)dt = \int g(t)dt + C.$$

The method of simple substitution can be applied to the integral on the left. If we substitute $u = x(t)$, then $du = (dx/dt)dt$, and the equation becomes

$$\int [1/h(u)]du = \int g(t)dt + C. \tag{2.5}$$

Now let $H(u)$ be any function such that $H'(u) = 1/h(u)$ and $G(t)$ any function with $G'(t) = g(t)$. Then (2.5) implies that

$$H(u) + C_1 = G(t) + C_2 \Longrightarrow H(u) = G(t) + C,$$

where C is the constant $C_2 - C_1$.

Replacing u again by $x(t)$:

$$H(x(t)) = G(t) + C. \tag{2.6}$$

Check carefully that the expression $H(x) = G(t) + C$ in (2.6) is exactly the same as the solution obtained in (2.3) in the above example. It is an **implicit solution** of (2.2); that is, it defines a relation between the unknown function x and its independent variable t. If it can be solved explicitly for x as a function of t, the result is called an **explicit solution** of the differential equation. As expected, the integration produces an infinite one-parameter family of solutions.

Everything that has been said so far justifies the following step-by-step procedure for solving separable equations.