Contents

CHAPTER 1

Introduction to Differential Equations

Differential equations arise from real-world problems and problems in applied mathematics. One of the first things you are taught in calculus is that the derivative of a function is the instantaneous rate of change of the function with respect to its independent variable. When mathematics is applied to real-world problems, it is often the case that finding a relation between a function and its rate of change is easier than finding a formula for the function itself; it is this relation between an unknown function and its derivatives that produces a differential equation.

To give a very simple example, a biologist studying the growth of a population, with size at time t given by the function $P(t)$, might make the very simple, but logical, assumption that a population grows at a rate directly proportional to its size. In mathematical notation, the equation for $P(t)$ could then be written as

$$\frac{dP}{dt} = rP(t),$$

where the constant of proportionality, r, would probably be determined experimentally by biologists working in the field. Equations used for modeling population growth can be much more complicated than this, sometimes involving scores of interacting populations with different properties; however, almost any population model is based on equations similar to this.

In an analogous manner, a physicist might argue that all the forces acting on a particular moving body at time t depend only on its position $x(t)$ and its velocity $x'(t)$. He could then use Newton's second law to express mass times acceleration as $mx''(t)$ and write an equation for $x(t)$ in the form

$$mx''(t) = F(x(t), x'(t)),$$

where F is some function of two variables. One of the best-known equations of this type is the spring-mass equation

$$mx'' + bx' + kx = f(t), \tag{1.1}$$

in which $x(t)$ is the position at time t of an object of mass m suspended on a spring, and b and k are the damping coefficient and spring constant, respectively. The function f represents an external force acting on the system. Notice that in (1.1), where x is a function of a single variable, we have used the convention of omitting the independent variable t, and have written x, x', and x'' for $x(t)$ and its derivatives.

In both of the examples, the problem has been written in the form of a differential equation, and the solution of the problem lies in finding a function $P(t)$, or $x(t)$, which makes the equation true.

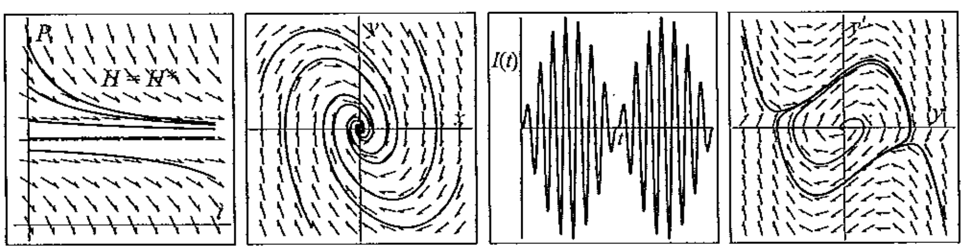
1.1 Basic Terminology

Before beginning to tackle the problem of formulating and solving differential equations, it is necessary to understand some basic terminology. Our first and most fundamental definition is that of a differential equation itself.

Definition 1.1. A **differential equation** is any equation involving an unknown function and one or more of its derivatives.

The following are examples of differential equations:

1.	$P'(t) = rP(t)(1 - P(t)/N) - H$	harvested population growth
2.	$\frac{d^2x}{d\tau^2} + 0.9\frac{dx}{d\tau} + 2x = 0$	spring-mass equation
3.	$I''(t) + 4I(t) = \sin(\omega t)$	RLC circuit showing "beats"
4.	$y''(t) + \mu(y^2(t) - 1)y'(t) + y(t) = 0$	van der Pol equation
5.	$\frac{\partial^2}{\partial x^2}u(x, y) + \frac{\partial^2}{\partial y^2}u(x, y) = 0$	Laplace's equation



For the first four equations the graphs above illustrate different ways of picturing the solution curves.

1.1.1 Ordinary vs. Partial Differential Equations

Differential equations fall into two very broad categories, called ordinary differential equations and partial differential equations. If the unknown function in the equation is a function of only one variable, the equation is called an **ordinary differential equation**. In the list of examples, equations 1-4 are ordinary differential equations, with the unknown functions being $P(t)$, $x(\tau)$, $I(t)$, and $y(t)$ respectively. If the unknown function in the equation depends on more than one independent variable, the equation is called a **partial differential equation**, and in this case, the derivatives appearing in the equation will be partial derivatives. Equation 5 is an example of an important partial differential equation, called Laplace's equation, which arises in several areas of applied mathematics. In equation 5, u is a function of the two independent variables x and y. In this book, we will not consider methods for solving partial differential equations. One of the basic methods involves reducing the partial differential equation to the solution of two or more ordinary differential equations, so it is important to have a solid grounding in ordinary differential equations first.

1.1.2 Independent Variables, Dependent Variables, and Parameters

Three different types of quantities can appear in a differential equation. The unknown function, for which the equation is to be solved, is called the **dependent variable**, and since we will be considering only ordinary differential equations, the dependent variable is a function of a single **independent variable**. In addition to the independent and dependent variables, a third type of variable, called a **parameter**, may appear in the equation. A parameter is a quantity that remains fixed in any specification of the problem, but can vary from problem to problem. In this book, parameters will usually be real numbers, such as r, N, and H in equation 1, ω in equation 3, and μ in equation 4.

1.1.3 Order of a Differential Equation

Another important way in which differential equations are classified is in terms of their order.

Definition 1.2. The **order** of a differential equation is the order of the highest derivative of the unknown function that appears in the equation.

The differential equation 1 is a first-order equation and the others are all second-order. Even though equation 5 is a partial differential equation, it is still said to be of second order since no derivatives (in this case partial derivatives) of order higher than two appear in the equation.

You may have noticed in the Table of Contents that some of the chapter headings refer to first-order or second-order differential equations. In some sense, first-order equations are thought of as being simpler than second-order equations. By the time you have worked through Chapter 2, you may not want to believe that this is true, and there are special cases where it definitely is not true; however, it is a useful way to distinguish between equations to which different methods of solution apply. In Chapter 4, we will see that solving ordinary differential equations of order greater than one can always be reduced to solving a system of first-order equations.

1.1.4 What is a Solution?

Given a differential equation, exactly what do we mean by a solution? It is first important to realize that we are looking for a function, and therefore it needs to be defined on some interval of its independent variable. Before computers were available, a solution of a differential equation usually referred to an analytic solution; that is, a formula obtained by algebraic methods or other methods of mathematical analysis such as integration and differentiation, from which exact values of the unknown function could be obtained.

Definition 1.3. An **analytic solution** of a differential equation is a sufficiently differentiable function that, if substituted into the equation, together with the necessary derivatives, makes the equation an identity (a true statement for all values of the independent variable) over some interval of the independent variable.

It is now possible, however, using sophisticated computer packages, to numerically approximate solutions to a differential equation to any desired degree of accuracy, even if no formula for the solution can be found. You will be introduced to numerical methods in Chapter 2, and many of the equations in later chapters will only be solvable using numerical or graphical methods.

Given an analytic solution, it is usually fairly easy to check whether or not it satisfies the equation. In Examples 1.1.1 and 1.1.2 a formula for the solution is given and you are only asked to verify that it satisfies the given differential equation.

Example 1.1.1. Show that the function $p(t) = e^{-2t}$ is a solution of the differential equation

$$x'' + 3x' + 2x = 0.$$

Solution To show that it is a solution, compute the first and second derivatives of $p(t)$:

$$p'(t) = -2e^{-2t}$$

$$p''(t) = 4e^{-2t}.$$

With the three functions $p(t)$, $p'(t)$, and $p''(t)$ substituted into the differential equation in place of x, x', and x'', it becomes

$$(4e^{-2t}) + 3(-2e^{-2t}) + 2(e^{-2t}) = (4 - 6 + 2)(e^{-2t}) = (0)(e^{-2t}) \equiv 0,$$

which is an identity (in the independent variable t) for all real values of t. ■

When showing that both sides of an equation are identical for all values of the variables, we will use the equivalence sign $\equiv$. This will be used as a convention throughout the book.

For practice, show that the function $q(t) = 3e^{-t}$ is also a solution of the equation $x'' + 3x' + 2x = 0$. It may seem surprising that two completely different functions satisfy this equation, but we will soon see that differential equations can have many solutions, in fact infinitely many. In the above example, the solutions p and q turned out to be functions that are defined for all real values of t. In the next example, things are not quite as simple.

Example 1.1.2. Show that the function $\phi(t) = (1 - t^2)^{1/2} \equiv \sqrt{1 - t^2}$ is a solution of the differential equation $x' = -t/x$.

Solution First, notice that $\phi(t)$ is not even defined outside of the interval $-1 \leq t \leq 1$. In the interval $-1 < t < 1$, $\phi(t)$ can be differentiated by the chain rule (for powers of functions):

$$\phi'(t) = (1/2)(1 - t^2)^{-1/2}(-2t) = -t/(1 - t^2)^{1/2}.$$

The right-hand side of the equation $x' = -t/x$, with $\phi(t)$ substituted for x, is

$$-t/\phi(t) = -t/(1 - t^2)^{1/2},$$

which is identically equal to $\phi'(t)$ wherever ϕ and ϕ' are both defined. Therefore, $\phi(t)$ is a solution of the differential equation $x' = -t/x$ on the interval $(-1, 1)$. ■

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2. $dP/dt = rP(1 -$
3. $dP/dt = rP(1 -$
4. $mx'' + bx' + kx$
5. $x''' + 2x'' + x' -$
6. $(ty'(t))' = \alpha e^t$
7. $d^2\theta/dt^2 + \sin(\theta$

You may be wondering if there are any solutions of $x' = -t/x$ that exist outside of the interval $-1 < t < 1$, since the differential equation is certainly defined outside of that interval. This problem will be revisited in Section 2.4 when we study the existence and uniqueness of solutions, and it will be shown that solutions do exist throughout the entire (t, x)-plane.

1.1.5 Systems of Differential Equations

In Chapter 4 we are going to study systems of differential equations, where two or more dependent variables are related to each other by differential equations. Linked equations of this sort appear in many real-world applications. As an example, ecologists studying the interaction between competing species in a particular ecosystem may find that the growth (think derivative, or rate of change) of each population can depend on the size of some or all of the other populations. To show that a set of formulas for the unknown populations is a solution of a system of this type, it must be shown that the functions, together with their derivatives, make every equation in the system an identity. The following simple example shows how this is done.

Example 1.1.3. Show that the functions

$$x(t) = e^{-t}, \quad y(t) = -4e^{-t}$$

form a solution of the system of differential equations

$$\begin{aligned} x'(t) &= 3x + y \\ y'(t) &= -4x - 2y. \end{aligned} \tag{1.2}$$

Solution The derivatives that we need are $x'(t) = -e^{-t}$ and $y'(t) = -(-4e^{-t}) = 4e^{-t}$. Then substitution into (1.2) gives

$$\begin{aligned} 3x + y &= 3(e^{-t}) + (-4e^{-t}) = (3-4)e^{-t} = -e^{-t} \equiv x'(t), \\ -4x - 2y &= -4(e^{-t}) - 2(-4e^{-t}) = (-4+8)e^{-t} = 4e^{-t} \equiv y'(t); \end{aligned}$$

therefore, the given functions for x and y form a solution for the system. ■

Exercises 1.1. For each equation 1–8 below, determine its order. Name the independent variable, the dependent variable, and any parameters in the equation.

1. $dy/dt = y^2 - t$
2. $dP/dt = rP(1 - P/k)$
3. $dP/dt = rP(1 - P/k) - \frac{\beta P^2}{\alpha^2 + P^2}$
4. $mx'' + bx' + kx = 2t^5$, assuming x is a function of t
5. $x''' + 2x'' + x' + 3x = \sin(\omega t)$, assuming x is a function of t
6. $(ty'(t))' = \alpha e^t$
7. $d^2\theta/dt^2 + \sin(\theta) = 4\cos(t)$

8. $y'' + \varepsilon(y^2 - 1)y' + y = 0$, assuming y is a function of t

For each equation 9–17 below, show that the given function is a solution. Determine the largest interval or intervals of the independent variable over which the solution is defined, and satisfies the equation.

9. $2x'' + 6x' + 4x = 0, \quad x(t) = e^{-2t}$
10. $x'' + 4x = 0, \quad x(t) = \sin(2t) + \cos(2t)$
11. $t^2x'' + 3tx' + x = 0, \quad x(t) = 1/t$
12. $t^2x'' + 3tx' + x = 0, \quad x(t) = \ln(t)/t$
13. $P' = rP, \quad P(t) = Ce^{rt}, \quad C$ any real number
14. $P' = rP(1 - P), \; P(t) = 1/(1 + Ce^{-rt}), \quad C$ any real number
15. $x' = (t + 2)/x; \quad x(t) = \sqrt{t^2 + 4t + 1}$
16. $x'' - 2tx' + 6x = 0, \quad x(t) = 8t^3 - 12t$
17. $t^2y'' + ty' + (t^2 - \frac{1}{4})y = 0, \quad y(t) = \frac{\sin(t)}{\sqrt{t}}$

In the next four problems, show that the given functions form a solution of the system. Determine the largest interval of the independent variable over which the solution is defined, and satisfies the equations.

18. System: $x' = x - y, \; y' = -4x + y$, solution is $x = e^{-t} - e^{3t}, \; y = 2e^{-t} + 2e^{3t}$.
19. System: $x' = x - y, \; y' = 4x + y$, solution is $x = e^t \cos 2t - \frac{1}{2}e^t \sin 2t, \; y = e^t \cos 2t + 2e^t \sin 2t$.
20. System: $x' = y, \; y' = 4x + 2y$, solution is $x = e^{-t}\sin(3t), \; y = e^{-t}(3\cos(3t) - \sin(3t))$.
21. System: $x' = x + 3y, \; y' = 4x + 2y$, solution is $x = 3e^{5t} + e^{-2t}, \; y = 4e^{5t} - e^{-2t}$.

1.2 Families of Solutions, Initial-value Problems

In this section the solutions of some very simple differential equations will be examined in order to give you an understanding of the terms n-parameter family of solutions and general solution of a differential equation. You will also be shown how to use certain types of information to pick one particular solution out of a set of solutions.

While you do not yet have any formal methods for solving differential equations, there are some very simple equations that can be solved by inspection. One of these is

$$x' = x. \tag{1.3}$$

This first-order differential equation asks you to find a function $x(t)$ which is equal to its own derivative at every value of t. Any calculus student knows one function that satisfies this property, namely the exponential function $x(t) = e^t$. In fact, one reason mathematicians use e as the basis of their exponential function is that e^t is the only function of the form a^t for which this is true (remember that in general, $\frac{d}{dt}(a^t) = a^t \ln(a)$, and $\ln(a) = 1$ only if $a = e$).

What may not be immediately clear is that the function

$$x(t) = Ce^t, \tag{1.4}$$

for any constant value of C, is also a solution of (1.3). Check it!

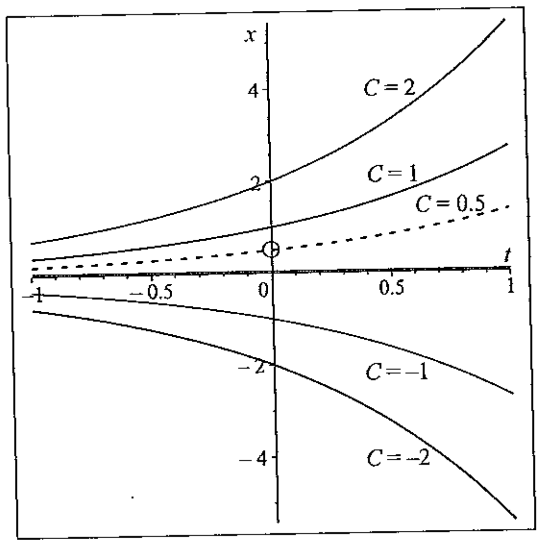

Figure 1.1. Curves in the one-parameter family $x = Ce^t$

This means that we have found an infinite set of solutions (1.4), called a **one-parameter family of solutions**, for the first order differential equation (1.3). In (1.4) the parameter is denoted by C. We will also be able to show later that this family contains every possible solution of (1.3), and when this is the case we call the one-parameter family (1.4) a **general solution** of the differential equation. In Figure 1.1 we have plotted $x(t) = Ce^t$ for several values of C, and you should be able to convince yourself that these curves are non-intersecting (do not cross), and fill up the (t, x)-plane.

To pick out a single solution curve it is only necessary to specify one point on the curve; that is, one **initial condition** of the form $x(t_0) = x_0$ must be given.

Definition 1.4. A first-order differential equation with one initial condition specified is called an **initial-value problem**, usually abbreviated as an **IVP**. The solution of an IVP will be called a **particular solution** of the differential equation.

Example 1.2.1. Solve the IVP

$$x' = x, \quad x(0) = \frac{1}{2}.$$

Solution Since we just found that the general solution of $x' = x$ is $x(t) = Ce^t$, we only need to use the initial condition to determine the value of C. This will pick out one particular curve in the family.

Substituting $t = 0$ and $x(0) = \frac{1}{2}$ into the general solution (1.4),

$$x(0) = Ce^0 = C = \frac{1}{2}.$$

With $C = \frac{1}{2}$, the solution of the IVP is $x(t) = \frac{1}{2}e^t$. This particular solution is the dotted curve shown in Figure 1.1, with the initial point $(0, \frac{1}{2})$ circled. ■

To see how this changes if the differential equation is of second order, consider the equation

$$x'' = -x. \tag{1.5}$$

There are two trigonometric functions, namely $\sin(t)$ and $\cos(t)$, that have the property that their second derivative is equal to the negative of the function. It is also easy to show that the function

$$x(t) = C_1 \sin(t) + C_2 \cos(t), \tag{1.6}$$

for any constants C_1 and C_2, satisfies (1.5). Check it! This time we have found a **two-parameter family of solutions** for the second-order equation, where the two parameters are denoted by C_1 and C_2. In Chapter 3 we will prove that *every* solution of (1.5) is of this form, and therefore (1.6) is again called a **general solution** of (1.5).

An **initial-value problem** for a second-order differential equation is going to require two initial conditions to determine the two constants C_1 and C_2.

To serve as **initial conditions**, it is necessary that the two conditions be the values of the function and its first derivative, both at the same value of t. If conditions are given at two different values of t, the problem is called a **boundary-value problem** and is, in general, much harder to solve. This problem will be encountered if you take a course on partial differential equations.

Example 1.2.2. Solve the IVP

$$x'' = -x, \quad x(0) = 2, \quad x'(0) = -1.$$

Solution The general solution was found to be

$$x(t) = C_1 \sin(t) + C_2 \cos(t).$$

We also need to have a formula for $x'(t)$, and this is obtained by differentiating $x(t)$:

$$x'(t) = C_1 \cos(t) - C_2 \sin(t).$$

If we use the two initial conditions, and let $t = 0$ in the formulas for x and x',

$$x(0) = C_1 \sin(0) + C_2 \cos(0) = C_1 \cdot 0 + C_2 \cdot 1 = C_2,$$

and

$$x'(0) = C_1 \cos(0) - C_2 \sin(0) = C_1 \cdot 1 - C_2 \cdot 0 = C_1.$$

Therefore, the values of the two constants must be $C_1 = x'(0) = -1$ and $C_2 = x(0) = 2$, and the solution of the IVP is uniquely determined to be

$$x(t) = -\sin(t) + 2\cos(t).$$ ■

Notice that we have no obvious way to picture the entire *two-parameter* family of solutions in the (t, x)-plane; much more will be said about this in Section 3.7 of Chapter 3 when the phase plane for a second-order differential equation is defined.

In summary, it will usually be the case that if an algebraic formula for a general solution of an nth order differential equation exists, it will contain n constants, and n initial conditions will be required to find a particular solution. It should be pointed out, however, that for the majority of differential equations, a simple formula for the general solution will not exist.

The equations $x' = x$ and $x'' = -x$ are examples of linear differential equations. This will be defined rigorously in Chapter 2, but for now it will mean that the right-hand side of the differential equation is a linear function of x and its derivatives; that is, a linear first-order equation must be of the form $x' = Ax + B$ where A and B can be functions of t, but not functions of x.

To see what can happen if the differential equation is not linear, let's look for solutions of the nonlinear first-order equation

$$x' = x^2. \tag{1.7}$$

It is not as easy to guess a solution for this equation, but in the first section of Chapter 2 you will be given a method for solving it. The function

$$x(t) = \frac{1}{C - t} \tag{1.8}$$

is a solution of (1.7) for any value of the constant C. Check it!

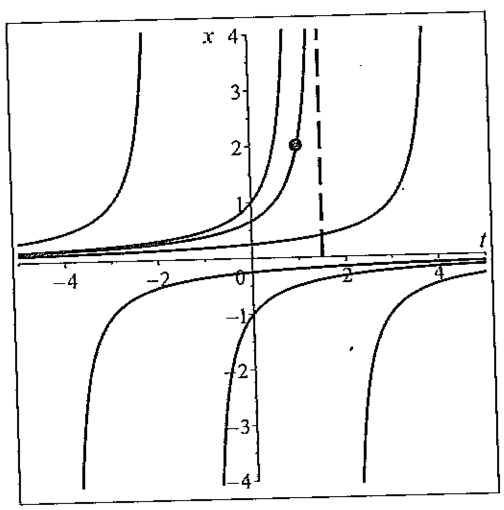

Figure 1.2. The curves $x(t) = \frac{1}{C-t}$

This one-parameter family of curves (1.8) is not quite a general solution; it does not contain the function $x(t) \equiv 0$, which obviously satisfies (1.7). However, the one-parameter family (1.8), together with the zero function does contain all solutions and can be considered a general solution. This family of curves is shown in Figure 1.2, plotted in the (t, x)-plane. It again appears to be a set that fills up the entire plane. For any initial condition