Name: ____________________________________

Lecture: ____________________________________ **Recitation:** ____________________________________

Multivariable Calculus

Math 215.70 Winter 2003

Final Exam

April 21, 2003

Please *do all your work in this booklet* and **show all the steps**.
Write your final answer *in the corresponding box.*
Calculators and note-cards are not allowed.

Problem	Possible points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	15	
7	10	
8	15	
9	10	
Total	100	

Problem 1. *(10 pts.)*

(a – 5 pts.) Find the area of the parallelogram spanned by the vectors $\mathbf{u} = \langle 3, 2, -1\rangle$ *and* $\mathbf{v} = \langle 1, -2, 3\rangle$.

(a)

Answer:

(b – 5 pts.) Find the rate of change of the function $f(x, y) = x^2y + y^3$ *at the point* $(2, 1)$ *in the direction of the vector* $\langle 3, -2\rangle$.

(b)

Answer:

	10

Problem 2. *(10 pts.) Three quantities* x, y, *and* z *are related by the equation* $x^3 z e^{z^2 - y^2} = 2$. *First find the equation of the tangent plane to the surface defined by this equation at the point* $(1, -2, 2)$ *and then use it to approximate the value of* z *corresponding to* $x = 1.2$ *and* $y = -2.1$.

Answer:

Problem 3. *(10 pts.) Find the mass of a cylindrical surface of radius* $r = 3$ *centered on the* z*-axis and bounded by the planes* $z = 1$ *and* $z = 3$ *if the density function is equal to the distance to the* xy*-plane.*

Answer:

	10
	10

Problem 4. *(10 pts.) Using any method you like (for example, the distance formula, Lagrange multipliers, etc.) find the distance from the point $A(17, -4, -3)$ to the point on the plane $6x - 3y + 2z = 10$ closest to A. What is this point?*

Answer:

10

Problem 5. *(10 pts.)*

The great Greek mathematician Archimedes *in his treatise* On Spirals† *gave the following definition of a spiral:*

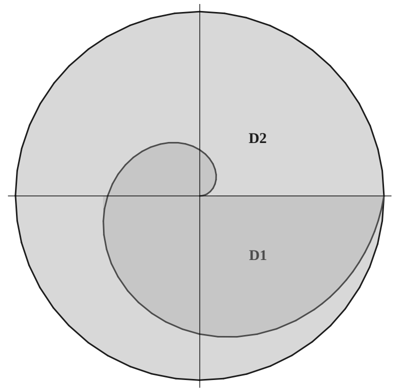
> *If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and if, at the same time as the line revolves, a point move at a uniform rate along the straight line, beginning from the extremity which remains fixed, the point will describe a spiral in the plane.*

Proposition 24 *in this treatise states that*

> *The area bounded by the first turn of the spiral and the initial line is equal to one-third of the 'first circle'.*

Archimedes proved it using the method of exhaustion.

Use integration in polar coordinates to give a simple proof of this Proposition*. Note that in polar coordinates the equation of the* Archimedian spiral *takes a simple form $r = a\theta$, where a is some constant. In the notations of the picture on the right, you have to show that $Area(D_1) = \frac{1}{3} Area(D_2)$ (note that D_1 is contained in D_2).*



† *The Works of Archimedes*, **Dover**, 1897

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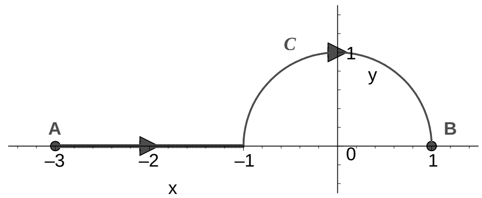
Problem 6. *(15 pts.)*

In this problem we consider two vector fields,

$$\mathbf{F} = \langle -y, x \rangle, \qquad \textit{and} \qquad \mathbf{G} = \langle \cos(x) + y, x - 1 \rangle,$$

and the curve C from the point $A(-3,0)$ to the point $B(1,0)$ that first goes along the x-axis, and then follows the unit circle (see the picture on the right).



(a – 5 pts.) Carefully explain whether these vector fields are conservative or not.

Answer: *(a)*

(b – 5 pts.) Compute the work done by the field $\mathbf{F}$ *along the curve* C.

Answer: *(b)*

(c – 5 pts.) Compute the work done by the field $\mathbf{G}$ *along the curve* C.

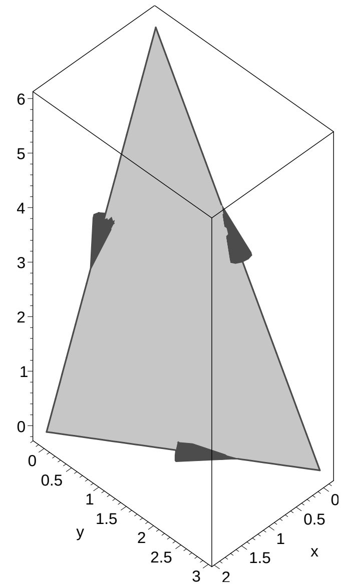
Answer: *(c)*

	15

Problem 7. *(10 pts.)*

Find the circulation of the vector field $\mathbf{F} = (4xy + xz)\mathbf{i} + (xy - yz)\mathbf{j} + (z^2 - xz)\mathbf{k}$ *along the curve* C *which is the boundary of the triangle with vertices* $A(2, 0, 0)$, $B(0, 3, 0)$, *and* $C(0, 0, 6)$ *oriented as shown.*

*(*Suggestion*: Use* ***Stokes' Theorem****).*



Answer:

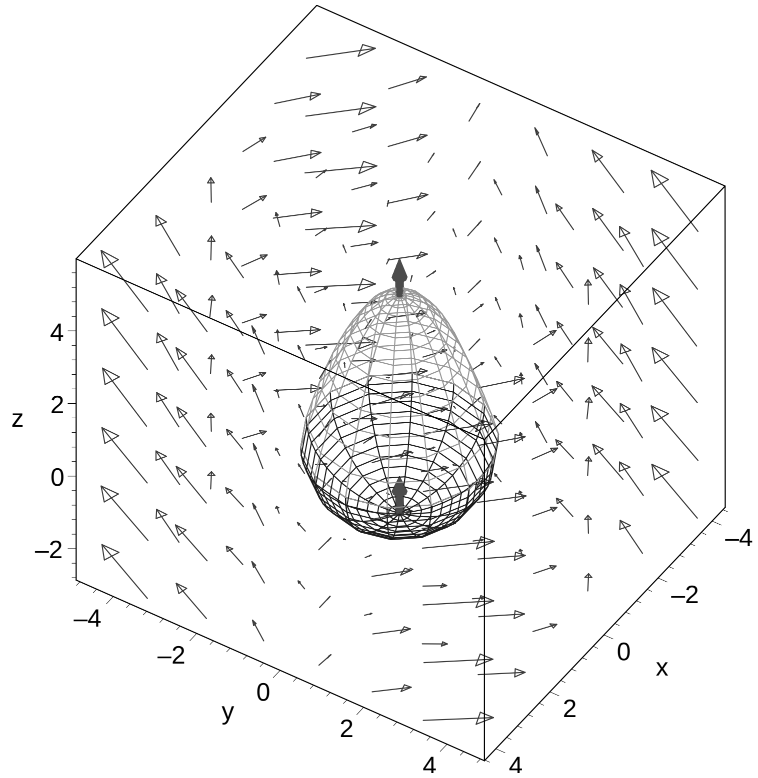
	10

Problem 8. *(15 pts.)*

*Use the **Divergence Theorem** to compare (by evaluating the difference) the flux of the vector field*

$$\mathbf{F} = (-y^2 + \cos(z))\mathbf{i} + (3xy + \sin(z))\mathbf{j} + (z + x^2)\mathbf{k}$$

through the surfaces S_1 and S_2, where S_1 is a lower hemisphere of radius 2 and S_2 is a part of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane and both surfaces are oriented upward (see the picture).



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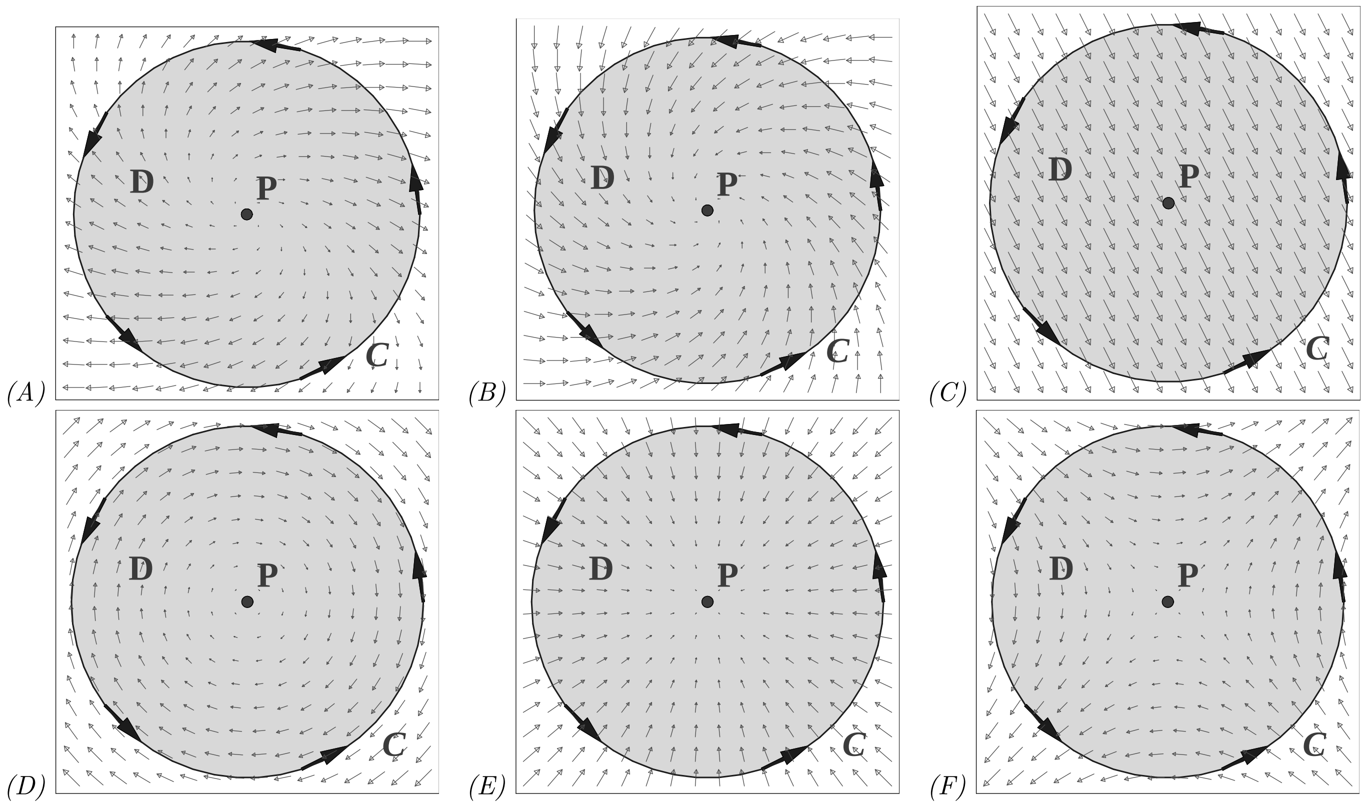
Problem 9. *(10 pts.) Below are six pictures of a vector field* $\mathbf{F}$*, region* D *and its oriented boundary* C*, and a point* P *inside* D*.*



For each of the properties below, indicate all plots that have that property.

(a) $\text{curl}(\mathbf{F})(P) > 0$ ____________________

(b) Circulation of $\mathbf{F}$ around C is positive ____________________

(c) Circulation of $\mathbf{F}$ around C is negative ____________________

(d) $\text{div}(\mathbf{F})(P) > 0$ ____________________

(e) Flux of $\mathbf{F}$ across C is negative ____________________

(f) $\mathbf{F}$ can be a gradient vector field ____________________

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