Math 2A – Vector Calculus – Chapter 13 Test – Fall ’09 Name____________________

Write all responses on separate paper. Show all work for credit. Don’t use a calculator.

1. Show that $\oint_C ye^z\,dx + xe^z\,dy + xye^z\,dz = 0$ for any closed curve *C*.

2. Consider $\vec{F} = \dfrac{\langle -y, x\rangle}{x^2+y^2}$ along the closed curve $C_a : x^2 + y^2 = a^2$

   a. Show that $\vec{\nabla} \times \vec{F} = \vec{0}$

   b. Compute $\oint_{C_a} \vec{F} \cdot \overrightarrow{dr}$.

   c. Explain why Stokes’ theorem doesn’t work in this case.

3. Compute the flux of the vector field $F = \langle x^3, 3xz^2, 3y^2z\rangle$ over the surface *S*, where *S* is boundary of the volume bounded by $z = 4 - x^2 - y^2$ and the *xy* plane.

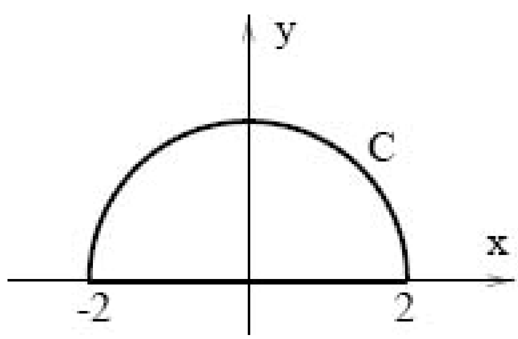
   a. Use the divergence theorem.

   b. Set up the surface integrals with simplified integrands and specific bounds of integration. Do not evaluate these integrals.

4. Compute the integral $\oint_C \vec{F} \cdot \overrightarrow{dr}$ where $\vec{F} = \langle y, x, x^2 + y^2\rangle$ and *C* is the positively oriented boundary curve of the part of the unit sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

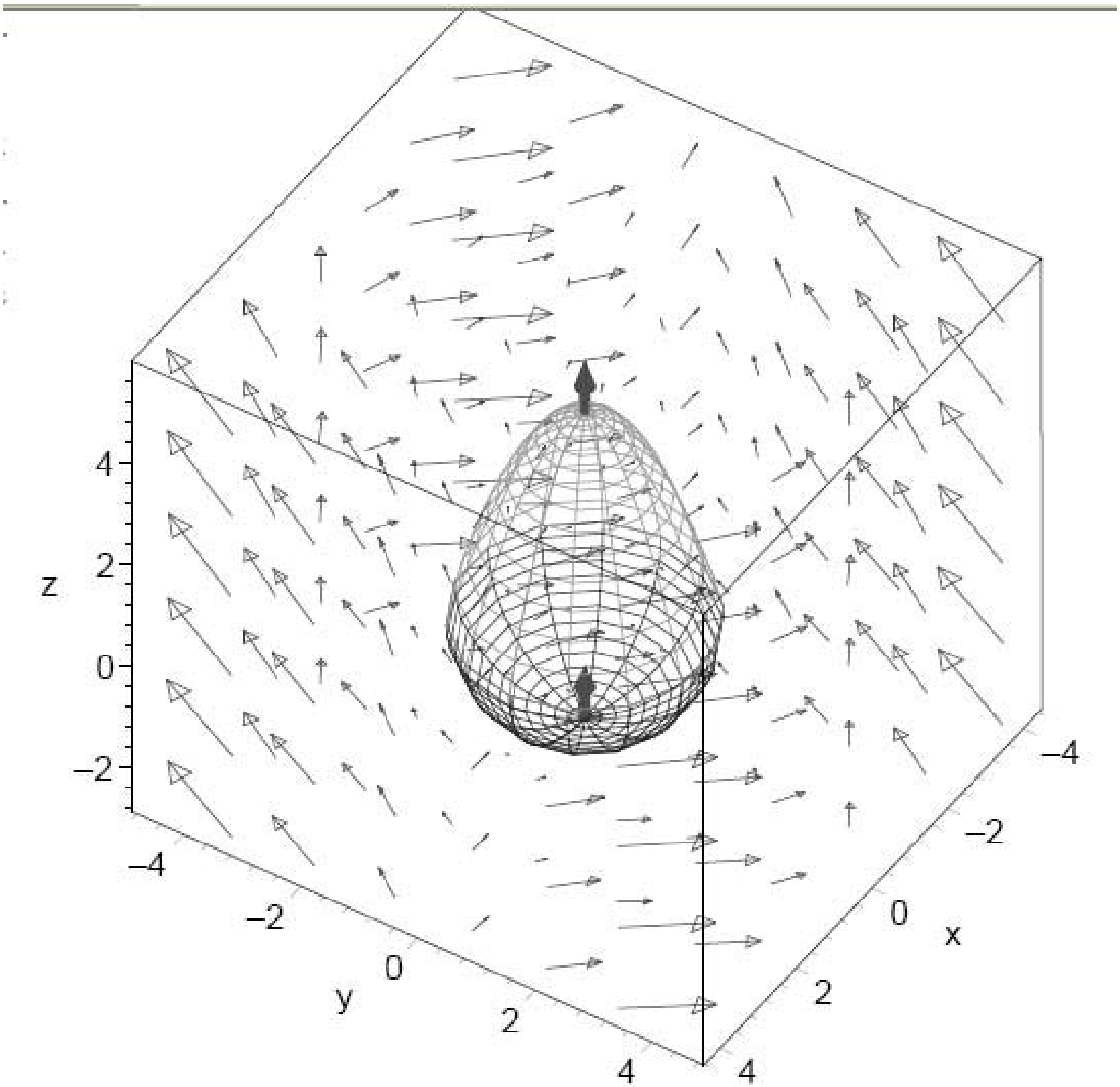
5. Assume that *f* and *g* are scalar functions on $\mathbb{R}^3$ with continuous second order partial derivatives. Prove that $\text{div}\left(\overrightarrow{\nabla f} \times \overrightarrow{\nabla g}\right) = 0$.

6. Let $\langle P, Q\rangle = \langle 4xy - 4y, 2x^2\rangle$. Use Green’s theorem to evaluate $\int_C \vec{F} \cdot \overrightarrow{dr}$ where *C* is the positively oriented curve that consists of the line segment from (–2,0) to (2,0) and the top half of the circle $x^2 + y^2 = 4$.



7. Compute the outward flux of $F(x, y, z) = \dfrac{\langle x, y, z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ through the ellipsoid $x^2 + y^2 + 4z^2 = 4$.

8. Use the divergence theorem to compare (by evaluating the difference) the flux of the vector field $F = \left\langle -y^2 + \cos(z), 3xz + \sin(z), z + x^2 \right\rangle$ through the surfaces S_1 and S_2 where S_1 is a lower hemisphere of radius 2 and S_2 is a part of the paraboloid $z = 4 - x^2 - y^2$ above the xy plane and both surfaces are oriented upward.



Math 2A – Vector Calculus – Chapter 13 Test Solutions – Fall '09

1. Show that $\oint_C ye^z\,dx + xe^z\,dy + xye^z\,dz = 0$ for any closed curve *C*.

   SOLN: $\vec{F} = \langle ye^z, xe^z, xye^z \rangle = \overrightarrow{\nabla f}$ where $f(x,y,z) = xye^z$, so $\vec{F}$ is a conservative vector field and so by the fundamental theorem of line integrals, the integral around any closed curve with by zero. A more advanced theorem you could also use here is that the curl of a vector field is everywhere zero if and only if the curl of the vector field is zero, that is the curl of a field is zero if and only if it's a conservative vector field. Here the curl of $\vec{F} = \langle ye^z, xe^z, xye^z \rangle$ is

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z & xye^z \end{vmatrix} = \langle xe^z - xe^z, ye^z - ye^z, e^z - e^z \rangle = \vec{0}$$

2. Consider $\vec{F} = \dfrac{\langle -y, x \rangle}{x^2 + y^2}$ along the closed curve $C_a : x^2 + y^2 = a^2$

   a. Show that $\vec{\nabla} \times \vec{F} = \vec{0}$

   SOLN: $$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = \left\langle 0, 0, \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} - \frac{-(x^2+y^2) + 2y^2}{(x^2+y^2)^2} \right\rangle$$

$$= \left\langle 0, 0, \frac{(y^2 - x^2) - (y^2 - x^2)}{(x^2+y^2)^2} \right\rangle = \vec{0}$$

   b. Compute $\oint_{C_a} \vec{F} \cdot \overrightarrow{dr}$.

   SOLN: Parameterize $\vec{r}(t) = \langle a\cos t, a\sin t \rangle$ so that $\vec{r}\,'(t) = \langle -a\sin t, a\cos t \rangle$ and

$$\vec{F} = \frac{\langle -a\sin t, a\cos t \rangle}{a^2\cos^2 t + a^2\sin^2 t} = \frac{\langle -\sin t, \cos t \rangle}{a} \text{ and so}$$

$$\oint_{C_a} \vec{F} \cdot \overrightarrow{dr} = \int_0^{2\pi} \frac{\langle -\sin t, \cos t \rangle}{a} \langle -a\sin t, a\cos t \rangle\, dt = \int_0^{2\pi} dt = 2\pi$$

c. Explain why Stokes' theorem doesn't work in this case.

SOLN: The vector field $\vec{F} = \frac{\langle -y, x\rangle}{x^2+y^2}$ doesn't have continuous partial derivatives at (0,0), a point in the region contained by C, so the conditions of the theorem are not met.

3. Compute the flux of the vector field $F = \langle x^3, 3xz^2, 3y^2z\rangle$ over the surface S, where S is boundary of the volume bounded by $z = 4 - x^2 - y^2$ and the xy plane.

a. Use the divergence theorem.

SOLN: $\oiint_S \vec{F}\cdot\overrightarrow{dS} = \iiint_V \vec{\nabla}\cdot\vec{F}dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{4-x^2-y^2} 3x^2+3y^2 dzdydx$ which lends itself nicely to polar coordinates: $\int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} 3r^2\left(rdzdrd\theta\right) = 2\pi\int_0^2 3r^2\left(4-r^2\right)dr = 2\pi\left(3r^4 - \frac{1}{2}r^6\right)\Big|_0^2 = 32\pi$

b. Set up the surface integrals with simplified integrands and specific bounds of integration. Do not evaluate these integrals.

SOLN: The surface of the paraboloid is parameterized by $\vec{r}\left(r,\theta\right) = \langle r\cos\theta, r\sin\theta, 4-r^2\rangle$ while the surface in the xy plane is simply $\vec{r}\left(x,y\right) = \langle x, y, 0\rangle$.

On the xy plane (flat bottom) the outward normal vector is $\overrightarrow{dS}_1 = \langle 0,0,-1\rangle$ and the Field vectors are $\langle x^3, 0, 0\rangle$ so $\vec{F}\cdot\overrightarrow{dS_1} = 0$. On the paraboloid,

$$\overrightarrow{dS}_2 = \left(\vec{r}_x \times \vec{r}_y\right)dA = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} dA = \langle 2x, 2y, 1\rangle dA \quad \text{so that}$$

$$\iint_{S_2}\left\langle x^3, 3x\left(4-x^2-y^2\right)^2, 3y^2\left(4-x^2-y^2\right)\right\rangle\cdot\langle 2x,2y,1\rangle dA$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2x^4 + 6xy\left(4-x^2-y^2\right)^2 + 3y^2\left(4-x^2-y^2\right)dydx$$

$$= \int_0^{2\pi}\int_0^2 2r^4\cos^4\theta + 6r^2\cos\theta\sin\theta\left(4-r^2\right)^2 + 3r^2\sin^2\theta\left(4-r^2\right)rdrd\theta$$

Try using a computer or calculator to verify this is 32π.

4. Compute the integral $\oint_C \vec{F}\cdot\overrightarrow{dr}$ where $\vec{F} = \langle y, x, x^2+y^2\rangle$ and C is the positively oriented boundary curve of the part of the unit sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

SOLN: This is a Stokes' theorem application. This surface can be parameterized by $\vec{r}\left(\phi,\theta\right) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$ so that

$$\overrightarrow{dS} = \left(\vec{r}_\phi \times \vec{r}_\theta\right) d\phi d\theta = \left\langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi \right\rangle d\phi d\theta$$

$$\vec{\nabla}\times\vec{F} = \vec{\nabla}\times\left\langle y, x, x^2+y^2\right\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x & x^2+y^2 \end{vmatrix} = \left\langle 2y, -2x, 0\right\rangle = \left\langle 2\sin\phi\sin\theta, -2\sin\phi\cos\theta, 0\right\rangle$$

$$\int_0^{\pi/2}\int_0^{\pi/2} \vec{\nabla}\times\vec{F}\cdot\overrightarrow{dS} = \int_0^{\pi/2}\int_0^{\pi/2} \left\langle 2\sin\phi\sin\theta, -2\sin\phi\cos\theta, 0\right\rangle \cdot \left\langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\right\rangle d\phi d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \left(2\sin^3\phi\sin\theta\cos\theta - 2\sin^3\phi\cos\theta\right) d\phi d\theta = 0$$

To be sure, we might what to check the line integral agrees with this. Along the curve C we have 3 ;ieces: $\vec{r}_1(t) = \langle \cos t, \sin t, 0\rangle$, $\vec{r}_2(t) = \langle 0, \cos t, \sin t\rangle$ and $\vec{r}_3(t) = \langle \sin t, 0, \cos t\rangle$ which is each traversed at t goes from 0 to $\pi/2$. Thus

$$\oint_C \vec{F}\cdot\overrightarrow{dr} = \int_0^{\pi/2} \left(\vec{F}(\vec{r}_1)\frac{d\vec{r}_1}{dt} + \vec{F}(\vec{r}_2)\frac{d\vec{r}_2}{dt} + \vec{F}(\vec{r}_3)\frac{d\vec{r}_3}{dt}\right) dt =$$

$$= \int_0^{\pi/2} \left(-\sin^2 t + \cos^2 t\right) + \left(\cos^3 t\right) + \left(-\sin^3 t\right) dt$$

$$= \int_0^{\pi/2} \left(\cos 2t + \cos^3 t - \sin^3 t\right) dt$$

$$= \frac{1}{2}\sin 2t\Big|_0^{\pi/2} + \frac{1}{3}\cos^2 t\sin t + \frac{2}{3}\sin t\Big|_0^{\pi/2} + \frac{1}{3}\left(2+\sin^2 t\right)\cos t\Big|_0^{\pi/2}$$

$$= 0 + \frac{2}{3} - \frac{2}{3} = 0$$

5. Assume that f and g are scalar functions on $\mathbb{R}^3$ with continuous second order partial derivatives. Prove that $\operatorname{div}\left(\overrightarrow{\nabla f}\times\overrightarrow{\nabla g}\right) = 0$.

SOLN:

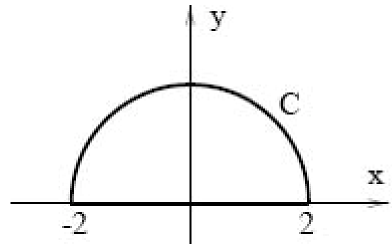
$$\vec{\nabla}\cdot\left(\vec{\nabla}f\times\vec{\nabla}g\right) = \vec{\nabla}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} = \vec{\nabla}\cdot\left(f_y g_z - f_z g_y, f_z g_x - f_x g_z, f_x g_y - f_y g_x\right)$$

$$= f_{yx}g_z + f_y g_{zx} - f_{zx}g_y - f_z g_{yx} + f_{zy}g_x + f_z g_{xy} - f_{xy}g_z - f_x g_{zy} + f_{xz}g_y + f_x g_{yz} - f_{yz}g_x - f_y g_{xz}$$

$$= \left(f_{yx}g_z - f_{xy}g_z\right) + \left(f_y g_{zx} - f_y g_{xz}\right) + \left(f_{xz}g_y - f_{zx}g_y\right)$$

$$+ \left(f_z g_{xy} - f_z g_{yx}\right) + \left(f_{zy}g_x - f_{yz}g_x\right) - f_x g_{zy} + f_x g_{yz}$$

6. Let $\langle P,Q\rangle = \langle 4xy-4y, 2x^2\rangle$. Use Green's theorem to evaluate $\int_C \vec{F}\cdot\overrightarrow{dr}$ where C is the positively oriented curve that consists of the line segment from (–2,0) to (2,0) and the top half of the circle $x^2 + y^2 = 4$.



SOLN: $\int_C \vec{F}\cdot\overrightarrow{dr} = \iint_D \left(Q_x - P_y\right)dA = \iint_D \left(4x-\left(4x-4\right)\right)dA = 4\left(\frac{\pi 2^2}{2}\right) = 8\pi$

7. Compute the outward flux of $F(x,y,z) = \frac{\langle x,y,z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}$ through the ellipsoid $x^2+y^2+4z^2=4$.

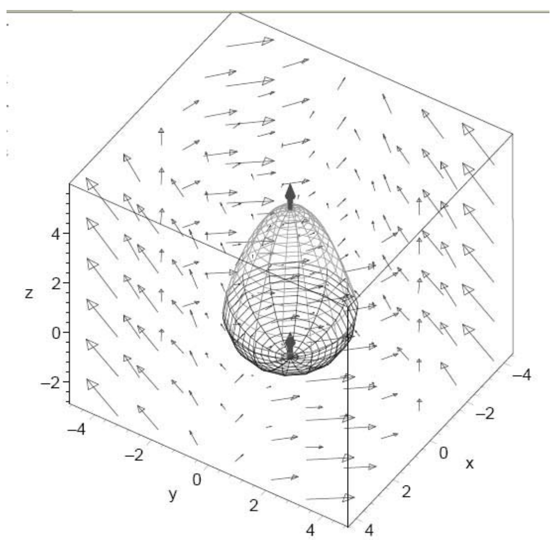
SOLN: $\oiint_S \vec{F}\cdot\overrightarrow{dS} = \iiint_E \vec{\nabla}\cdot\vec{F}dV$ where

$$\vec{\nabla}\cdot\vec{F} = \frac{x^2+y^2+z^2-3x^2}{\left(x^2+y^2+z^2\right)^{5/2}} + \frac{x^2+y^2+z^2-3y^2}{\left(x^2+y^2+z^2\right)^{5/2}} + \frac{x^2+y^2+z^2-3z^2}{\left(x^2+y^2+z^2\right)^{5/2}} = 0$$

So $\oiint_S \vec{F}\cdot\overrightarrow{dS} = 0$

8. Use the divergence theorem to compare (by evaluating the difference) the flux of the vector field $F = \langle -y^2+\cos(z), 3xz+\sin(z), z+x^2\rangle$ through the surfaces S_1 and S_2 where S_1 is a lower hemisphere of radius 2 and S_2 is a part of the paraboloid $z = 4 - x^2 - y^2$ above the xy plane and both surfaces are oriented upward.

SOLN: The plan is to compute the outward flux through the volume above the plane and the inward flux through the volume below the plane. This counts the downward flux through the circular cross section of the xy-plane twice, so we may need to adjust for that, unless it's zero.



So the outward flux of the upper half is

$$\oiint_S \vec{F}\cdot\overrightarrow{dS} = \iiint_V \left(\vec{\nabla}\cdot\vec{F}\right)dV = \iiint_V 1dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} rdzdrd\theta = 2\pi\int_0^2 r\left(4-r^2\right)dr = 2\pi\left(2r^2-\frac{r^3}{3}\right)\Big|_0^2 = \frac{32\pi}{3}$$

…and the inward flux of the lower half is xz

$-\iiint_V 1dV = -\frac{4\pi(2)^3}{3} = -\frac{32\pi}{3}$. Well, now for double the upward flux through the disk in the *xy*-plane:

$$2\iint_D \vec{F}\cdot\overrightarrow{dS} = 2\iint_D \left\langle y^2+1, 0, x^2\right\rangle\cdot\langle 0,0,1\rangle dS = 2\iint_D x^2 dA = 2\int_0^{2\pi}\int_0^2 r^3\cos^2\theta dr d\theta$$

$$= 2\int_0^{2\pi}\cos^2\theta d\theta\int_0^2 r^3 dr = 4\pi$$

I think that's right. I dare you to prove me wrong.