CLASSROOM NOTES

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VECTOR TRIPLE PRODUCT

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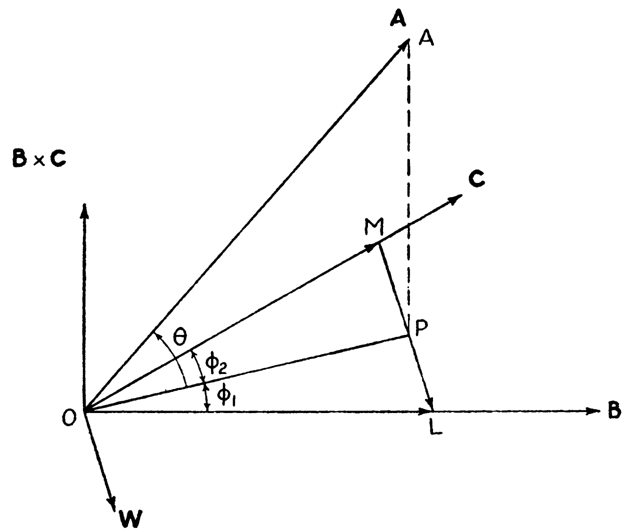
The standard expression for the vector triple product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ in terms of the vectors $\mathbf{b}$ and $\mathbf{c}$ is usually obtained by a quite satisfactory analytical procedure. (See: *Vector and Tensor Analysis*, Brand; New York, 1947, p. 40.) However a direct geometrical derivation might be of more appeal to some minds and perhaps also would be more in keeping with the spirit of vector algebra. It is the purpose of this note to present a direct geometrical construction for this product. (For another geometrical treatment, see Coe and Rainich, this MONTHLY, vol. 56, 1949, p. 175.)



First note that the above product can be written as

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = abc\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$$

where a, b, c are the magnitudes of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are unit vectors along $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. We are thus concerned with the problem of finding an expression for the vector $\mathbf{w}=\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. Let these vectors in one of several possible configurations be as shown in the diagram, starting at O, with $\mathbf{B}$ and $\mathbf{C}$ in a horizontal plane. The vector product $\mathbf{B}\times\mathbf{C}$ will be a vertical vector of magnitude $\sin\phi$ where $\phi=\angle(\mathbf{B},\mathbf{C})$. Let P be the projection on the plane of $\mathbf{B}$ and $\mathbf{C}$ of the end point A of $\mathbf{A}$. The triple product $\mathbf{w}$ will then be in the plane of **B** and **C**, perpendicular to OP as shown, and of magnitude

$$|\mathbf{w}| = \sin\phi\cos\theta = OP\sin\phi$$

where $\theta = \angle(OP, \mathbf{A})$.

Draw through P a line perpendicular to OP, intersecting the vectors **C** and **B** at M and L respectively. Then, equating the two expressions following for the area of the triangle OML, we get

$$\tfrac{1}{2}OP\cdot ML = \tfrac{1}{2}OM\cdot OL\sin\phi,$$

whence

$$\tfrac{1}{2}OP\cdot ML = \tfrac{1}{2}\overline{OP}^2\sin\phi\sec\phi_1\sec\phi_2$$

where $\phi_1 = \angle LOP$ and $\phi_2 = \angle POM$.
We thus obtain

$$ML = OP\sin\phi\sec\phi_1\sec\phi_2 = |\mathbf{w}|\sec\phi_1\sec\phi_2$$

and since $\overrightarrow{ML}$ and **w** are parallel

$$\mathbf{w} = \cos\phi_1\cos\phi_2\overrightarrow{ML}.$$

Now $\overrightarrow{ML} = \overrightarrow{OL} - \overrightarrow{OM} = OP\sec\phi_1\mathbf{B} - OP\sec\phi_2\mathbf{C}$ so that

$$\mathbf{w} = OP\cos\phi_2\mathbf{B} - OP\cos\phi_1\mathbf{C}.$$

Next we observe, since $\mathbf{A} = \overrightarrow{OP} + \overrightarrow{PA}$, that

$$\mathbf{A}\cdot\mathbf{B} = \overrightarrow{OP}\cdot\mathbf{B} + \overrightarrow{PA}\cdot\mathbf{B} = OP\cos\phi_1,$$

and similarly, that

$$\mathbf{A}\cdot\mathbf{C} = OP\cos\phi_2.$$

We thus obtain the expansion for the product **w**,

$$\mathbf{w} = \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C},$$

and finally we get the standard expansion for the vector triple product

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = abc\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}.$$

It may be remarked here that the establishment of the distributive law for the vector product, say $(\mathbf{a}+\mathbf{b})\times\mathbf{u}$ can be made to depend on the distributive law for the scalar product by replacing **u** by the vector product of two appropriate vectors and then using the above standard expansion for the vector triple product.