15.6 Stokes' Theorem and the Curl of F

For the Divergence Theorem, the surface was closed. S was the boundary of V. Now the surface is not closed and S has its own boundary—a curve called C. We are back near the original setting for Green's Theorem (region bounded by curve, double integral equal to work integral). But Stokes' Theorem, also called Stokes's Theorem, is in three-dimensional space. There is a ***curved surface*** S bounded by a ***space curve*** C. This is our first integral around a space curve.

The move to three dimensions brings a change in the vector field. The plane field $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ becomes a space field $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$. The work $M\,dx + N\,dy$ now includes $P\,dz$. The critical quantity in the double integral (it was $\partial N/\partial x - \partial M/\partial y$) must change too. We called this scalar quantity "curl **F**," but in reality it is only the third component of a vector. Stokes' Theorem needs all three components of that vector—which is curl **F**.

DEFINITION The curl of a vector field $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is the vector field

$$\text{curl }\mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}. \tag{1}$$

The symbol $\nabla \times \mathbf{F}$ stands for a determinant that yields those six derivatives:

$$\text{curl }\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}. \tag{2}$$

The three products $\mathbf{i}\ \partial/\partial y\ P$ and $\mathbf{j}\ \partial/\partial z\ M$ and $\mathbf{k}\ \partial/\partial x\ N$ have plus signs. The three products like $\mathbf{k}\ \partial/\partial y\ M$, down to the left, have minus signs. There is a cyclic symmetry. This determinant helps the memory, even if it looks and is illegal. A determinant is not supposed to have a row of vectors, a row of operators, and a row of functions.

EXAMPLE 1 The plane field $M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ has $P = 0$ and $\partial M/\partial z = 0$ and $\partial N/\partial z = 0$. Only two terms survive: curl $\mathbf{F} = (\partial N/\partial x - \partial M/\partial y)\mathbf{k}$. Back to Green.

EXAMPLE 2 The cross product $\mathbf{a} \times \mathbf{R}$ is a ***spin field*** **S**. Its axis is the fixed vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. The flow in Figure 15.23 turns around **a**, and its components are

$$\mathbf{S} = \mathbf{a} \times \mathbf{R} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y)\mathbf{i} + (a_3 x - a_1 z)\mathbf{j} + (a_1 y - a_2 x)\mathbf{k}. \tag{3}$$

Our favorite spin field $-y\mathbf{i} + x\mathbf{j}$ has $(a_1, a_2, a_3) = (0,0,1)$ and its axis is $\mathbf{a} = \mathbf{k}$.

The divergence of a spin field is $M_x + N_y + P_z = 0 + 0 + 0$. Note how the divergence uses M_x while the curl uses N_x and P_x. ***The curl of* S *is the vector*** $2\mathbf{a}$:

$$\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = 2a_1\mathbf{i} + 2a_2\mathbf{j} + 2a_3\mathbf{k} = 2\mathbf{a}.$$

This example begins to reveal the meaning of the curl. It measures the spin! The direction of curl **F** is the ***axis of rotation***—in this case along **a**. The magnitude of curl **F** is ***twice the speed of rotation***. In this case $|\text{curl}\,\mathbf{F}| = 2|\mathbf{a}|$ and the angular velocity is $|\mathbf{a}|$.

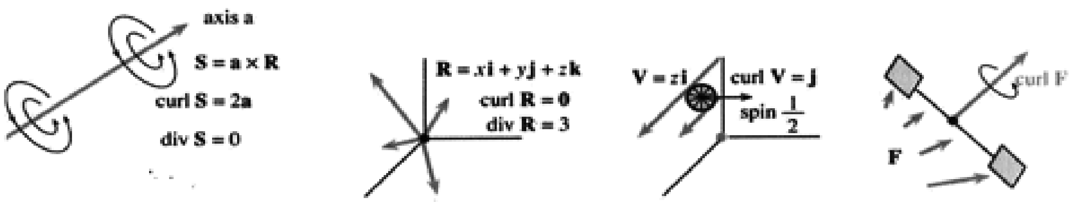

Fig. 15.23 Spin field $\mathbf{S} = \mathbf{a} \times \mathbf{R}$, position field **R**, velocity field (shear field)$\mathbf{V} = z\mathbf{i}$, any field **F**.

EXAMPLE 3 (!!) *Every gradient field* $\mathbf{F} = \partial f/\partial x\ \mathbf{i} + \partial f/\partial y\ \mathbf{j} + \partial f/\partial z\ \mathbf{k}$ *has* curl $\mathbf{F} = \mathbf{0}$:

$$\text{curl}\,\mathbf{F} = \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\right)\mathbf{i} + \left(\frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z}\right)\mathbf{j} + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right)\mathbf{k} = \mathbf{0}. \quad (4)$$

Always f_{yz} equals f_{zy}. They cancel. Also $f_{xz} = f_{zx}$ and $f_{yx} = f_{xy}$. So curl grad $f = \mathbf{0}$.

EXAMPLE 4 (twin of Example 3) The divergence of curl **F** is also automatically zero:

$$\text{div curl}\,\mathbf{F} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = 0. \quad (5)$$

Again the mixed derivatives give $P_{xy} = P_{yx}$ and $N_{xz} = N_{zx}$ and $M_{zy} = M_{yz}$. The terms cancel in pairs. In "curl grad" and "div curl", everything is arranged to give zero.

> **15N** The curl of the gradient of every $f(x,y,z)$ is curl grad $f = \nabla \times \nabla f = \mathbf{0}$.
> The divergence of the curl of every $\mathbf{F}(x,y,z)$ is div curl $\mathbf{F} = \nabla \cdot \nabla \times \mathbf{F} = 0$.

The spin field **S** has no divergence. The position field **R** has no curl. **R** is the gradient of $f = \frac{1}{2}(x^2+y^2+z^2)$. **S** is the curl of a suitable **F**. Then div $\mathbf{S} =$ div curl **F** and curl $\mathbf{R} =$ curl grad f are automatically zero.

You correctly believe that curl **F** measures the "spin" of the field. You may expect that curl $(\mathbf{F}+\mathbf{G})$ is curl $\mathbf{F}$ + curl **G**. Also correct. Finally you may think that a field of parallel vectors has no spin. That is wrong. Example 5 has parallel vectors, but their different lengths produce spin.

EXAMPLE 5 The field $\mathbf{V} = z\mathbf{i}$ in the x direction has curl $\mathbf{V} = \mathbf{j}$ in the y direction.

If you put a wheel in the xz plane, ***this field will turn it***. The velocity $z\mathbf{i}$ at the top of the wheel is greater than $z\mathbf{i}$ at the bottom (Figure 15.23c). So the top goes faster and the wheel rotates. The axis of rotation is curl $\mathbf{V} = \mathbf{j}$. The turning speed is $\frac{1}{2}$, because this curl has magnitude 1.

Another velocity field $\mathbf{v} = -x\mathbf{k}$ produces the same spin: curl $\mathbf{v} = \mathbf{j}$. The flow is in the z direction, it varies in the x direction, and the spin is in the y direction. Also interesting is $\mathbf{V}+\mathbf{v}$. The two "shear fields" add to a perfect spin field $\mathbf{S} = z\mathbf{i} - x\mathbf{k}$, whose curl is $2\mathbf{j}$.

THE MEANING OF CURL F

Example 5 put a paddlewheel into the flow. This is possible for any vector field **F**, and it gives insight into curl **F**. The turning of the wheel (if it turns) depends on its location (x, y, z). The turning also depends on the *orientation* of the wheel. We could put it into a spin field, and if the wheel axis **n** is perpendicular to the spin axis **a**, the wheel won't turn! The general rule for turning speed is this: ***the angular velocity of the wheel is*** $\frac{1}{2}(\text{curl }\mathbf{F})\cdot\mathbf{n}$. This is the "*directional spin*," just as $(\text{grad } f)\cdot\mathbf{u}$ was the "*directional derivative*"—and **n** is a unit vector like **u**.

There is no spin anywhere in a gradient field. It is ***irrotational***: curl grad $f = \mathbf{0}$.

The pure spin field $\mathbf{a}\times\mathbf{R}$ has curl $\mathbf{F} = 2\mathbf{a}$. The angular velocity is $\mathbf{a}\cdot\mathbf{n}$ (note that $\frac{1}{2}$ cancels 2). This turning is everywhere, ***not just at the origin***. If you put a penny on a compact disk, it turns once when the disk rotates once. That spin is "*around itself*," and it is the same whether the penny is at the center or not.

The turning speed is greatest when the wheel axis **n** lines up with the spin axis **a**. Then $\mathbf{a}\cdot\mathbf{n}$ is the full length $|\mathbf{a}|$. The gradient gives the direction of fastest growth, and the curl gives the direction of fastest turning:

maximum growth rate of f is $|\text{grad } f|$ in the direction of grad f

maximum rotation rate of **F** is $\frac{1}{2}|\text{curl }\mathbf{F}|$ in the direction of curl **F**.

STOKES' THEOREM

Finally we come to the big theorem. It will be like Green's Theorem—a line integral equals a surface integral. The line integral is still the work $\oint \mathbf{F}\cdot d\mathbf{R}$ around a curve. The surface integral in Green's Theorem is $\iint (N_x - M_y)\,dx\,dy$. The surface is flat (in the xy plane). Its normal direction is **k**, and we now recognize $N_x - M_y$ as the **k** component of the curl. Green's Theorem uses only this component because the normal direction is always **k**. For Stokes' Theorem on a curved surface, we need all three components of curl **F**.

Figure 15.24 shows a hat-shaped surface S and its boundary C (a closed curve). Walking in the positive direction around C, with your head pointing in the direction of **n**, the surface is *on your left*. You may be standing straight up ($\mathbf{n} = \mathbf{k}$ in Green's Theorem). You may even be upside down ($\mathbf{n} = -\mathbf{k}$ is allowed). In that case you must go the other way around C, to keep the two sides of equation (6) equal. The surface is still on the left. A Möbius strip is not allowed, because its normal direction cannot be established. The unit vector **n** determines the "counterclockwise direction" along C.

15O ***(Stokes' Theorem)*** $$\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS. \qquad (6)$$

The right side adds up small spins in the surface. The left side is the total circulation (or work) around C. That is not easy to visualize—this may be the hardest theorem in the book—but notice one simple conclusion. ***If*** curl $\mathbf{F} = \mathbf{0}$ ***then*** $\oint \mathbf{F}\cdot d\mathbf{R} = 0$. ***This applies above all to gradient fields***—as we know.

A gradient field has no curl, by (4). A gradient field does no work, by (6). In three dimensions as in two dimensions, ***gradient fields are conservative fields***. They will be the focus of this section, after we outline a proof (or two proofs) of Stokes' Theorem.

The first proof shows *why* the theorem is true. The second proof shows that it really is true (and how to compute). You may prefer the first.

First proof Figure 15.24 has a triangle ABC attached to a triangle ACD. Later there can be more triangles. S will be ***piecewise flat***, close to a curved surface. Two triangles are enough to make the point. In the plane of each triangle (they have different $\mathbf{n}$'s) Green's Theorem is known:

$$\oint_{AB+BC+CA} \mathbf{F}\cdot d\mathbf{R} = \iint_{ABC} \text{curl}\,\mathbf{F}\cdot\mathbf{n}dS \qquad \oint_{AC+CD+DA} \mathbf{F}\cdot d\mathbf{R} = \iint_{ACD} \text{curl}\,\mathbf{F}\cdot\mathbf{n}dS.$$

Now add. The right sides give $\iint \text{curl}\,\mathbf{F}\cdot\mathbf{n}dS$ over the two triangles. On the left, ***the integral over CA cancels the integral over AC***. The "crosscut" disappears. That leaves $AB+BC+CD+DA$. This line integral goes around the outer boundary C—which is the left side of Stokes' Theorem.

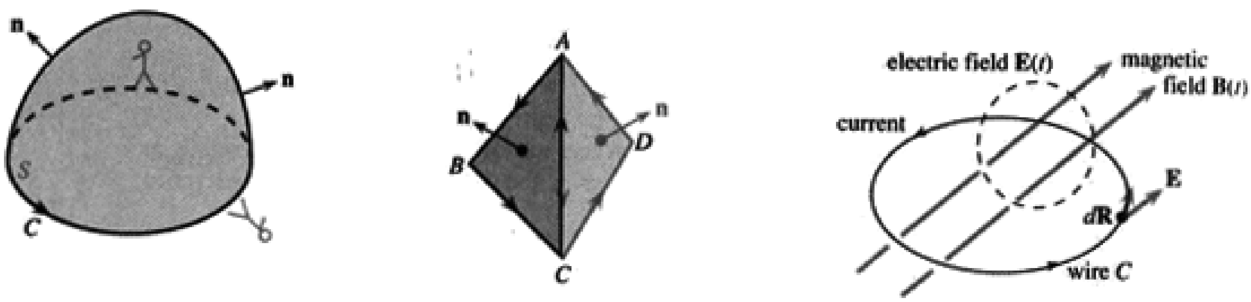

Fig. 15.24 Surfaces S and boundary curves C. Change in $\mathbf{B}\to$ curl $\mathbf{E}\to$ current in C.

Second proof Now the surface can be curved. A new proof may seem excessive, but it brings formulas you could compute with. From $z=f(x,y)$ we have

$$dz = \partial f/\partial x\, dx + \partial f/\partial y\, dy \quad \text{and} \quad \mathbf{n}dS = (-\partial f/\partial x\,\mathbf{i} - \partial f/\partial y\,\mathbf{j}+\mathbf{k})dx\,dy.$$

For $\mathbf{n}dS$, see equation 15.4.11. With this dz, the line integral in Stokes' Theorem is

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \oint_{\text{shadow of C}} M dx + N dy + P(\partial f/\partial x\, dx + \partial f/\partial y\, dy). \qquad (7)$$

The dot product of curl $\mathbf{F}$ and $\mathbf{n}dS$ gives the surface integral $\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}dS$:

$$\iint_{\text{shadow of S}} [(P_y - N_z)(-\partial f/\partial x) + (M_z - P_x)(-\partial f/\partial y) + (N_x - M_y)]dx\,dy. \quad (8)$$

To prove (7) = (8), change M in Green's Theorem to $M + P\partial f/\partial x$. Also change N to $N + P\partial f/\partial y$. Then (7) = (8) is Green's Theorem down on the shadow (Problem 47). This proves Stokes' Theorem up on S. Notice how Green's Theorem (flat surface) was the key to both proofs of Stokes' Theorem (curved surface).

EXAMPLE 6 Stokes' Theorem in electricity and magnetism yields Faraday's Law.

Stokes' Theorem is not heavily used for calculations—equation (8) shows why. But the spin or curl or ***vorticity*** of a flow is absolutely basic in fluid mechanics. The other important application, coming now, is to electric fields. Faraday's Law is to Gauss's Law as Stokes' Theorem is to the Divergence Theorem.

Suppose the curve C is an actual wire. We can produce current along C by varying the magnetic field $\mathbf{B}(t)$. The flux $\varphi = \iint \mathbf{B}\cdot\mathbf{n}\,dS$, passing within C and changing in time, creates an electric field $\mathbf{E}$ that does work:

$$\textit{Faraday's Law} \text{ (integral form)}: \text{work} = \oint_C \mathbf{E}\cdot d\mathbf{R} = -d\varphi/dt.$$

That is physics. It may be true, it may be an approximation. Now comes mathematics (surely true), which turns this integral form into a differential equation. Information at points is more convenient than information around curves. Stokes converts the line integral of $\mathbf{E}$ into the surface integral of curl $\mathbf{E}$:

$$\oint_C \mathbf{E}\cdot d\mathbf{R} = \iint_S \text{curl } \mathbf{E}\cdot\mathbf{n}\,dS \text{ and also } -\partial\varphi/\partial t = \iint_S -(\partial\mathbf{B}/\partial t)\cdot\mathbf{n}\,dS.$$

These are equal for any curve C, however small. So the right sides are equal for any surface S. We squeeze to a point. The right hand sides give one of Maxwell's equations:

$$\textit{Faraday's Law} \text{(differential form): curl } \mathbf{E} = -\partial\mathbf{B}/\partial t.$$

CONSERVATIVE FIELDS AND POTENTIAL FUNCTIONS

The chapter ends with our constant and important question: Which fields do no work around closed curves? Remember test **D** for plane curves and plane vector fields:

$$\text{if } \partial M/\partial y = \partial N/\partial x \text{ then } \mathbf{F} \text{ is conservative and } \mathbf{F} = \text{ grad } f \text{ and } \oint \mathbf{F}\cdot d\mathbf{R} = 0.$$

Now allow a three-dimensional field like $\mathbf{F} = 2xy\,\mathbf{i} + (x^2+z)\mathbf{j} + y\mathbf{k}$. Does it do work around a space curve? *Or is it a gradient field*? That will require $\partial f/\partial x = 2xy$ and $\partial f/\partial y = x^2+z$ and $\partial f/\partial z = y$. ***We have three equations for one function*** $f(x,y,z)$. Normally they can't be solved. When test **D** is passed (now it is the three-dimensional test: curl $\mathbf{F} = \mathbf{0}$) they *can* be solved. This example passes test **D**, and f is $x^2y + yz$.

> **15P** $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a conservative field if it has these properties:
>
> **A.** The work $\oint \mathbf{F}\cdot d\mathbf{R}$ around every closed path in space is zero.
>
> **B.** The work $\int_P^Q \mathbf{F}\cdot d\mathbf{R}$ depends on P and Q, not on the path in space.
>
> **C.** $\mathbf{F}$ is a ***gradient field***: $M = \partial f/\partial x$ and $N = \partial f/\partial y$ and $P = \partial f/\partial z$.
>
> **D.** The components satisfy $M_y = N_x$, $M_z = P_x$, and $N_z = P_y$ (curl **F** ***is zero***).
>
> A field with one of these properties has them all. **D** is the quick test.

A detailed proof of $\mathbf{A} \Rightarrow \mathbf{B} \Rightarrow \mathbf{C} \Rightarrow \mathbf{D} \Rightarrow \mathbf{A}$ is not needed. Only notice how $\mathbf{C} \Rightarrow \mathbf{D}$: curl grad $\mathbf{F}$ is always zero. The newest part is $\mathbf{D} \Rightarrow \mathbf{A}$. ***If*** curl $\mathbf{F} = \mathbf{0}$ ***then*** $\oint \mathbf{F}\cdot d\mathbf{R} = 0$. But that is not news. It is Stokes' Theorem.

The interesting problem is to solve the three equations for f, when test **D** is passed. The example above had

$$\partial f/\partial x = 2xy \Rightarrow f = \int 2xy\,dx = x^2y \text{ plus any function } C(y,z)$$
$$\partial f/\partial y = x^2 + z = x^2 + \partial C/\partial y \Rightarrow C = yz \text{ plus any function } c(z)$$
$$\partial f/\partial z = y = y + dc/dz \Rightarrow c(z) \text{ can be zero.}$$

The first step leaves an arbitrary $C(y,z)$ to fix the second step. The second step leaves an arbitrary $c(z)$ to fix the third step (not needed here). Assembling the three steps, $f = x^2y + C = x^2y + yz + c = x^2y + yz$. Please recognize that the "fix-up" is only possible when curl $\mathbf{F} = \mathbf{0}$. Test **D** must be passed.

EXAMPLE 7 Is $\mathbf{F} = (z-y)\mathbf{i} + (x-z)\mathbf{j} + (y-x)\mathbf{k}$ the gradient of any f?

Test **D** says *no*. This **F** is a spin field $\mathbf{a} \times \mathbf{R}$. Its curl is $2\mathbf{a} = (2,2,2)$, which is not zero. A search for f is bound to fail, but we can try. To match $\partial f/\partial x = z - y$, we must have $f = zx - yx + C(y,z)$. The y derivative is $-x + \partial C/\partial y$. That never matches $N = x - z$, so f can't exist.

EXAMPLE 8 What choice of P makes $\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + P\mathbf{k}$ conservative? Find f.

Solution We need curl $\mathbf{F} = 0$, by test **D**. First check $\partial M/\partial y = z^2 = \partial N/\partial x$. Also

$$\partial P/\partial x = \partial M/\partial z = 2yz \quad \text{and} \quad \partial P/\partial y = \partial N/\partial z = 2xz.$$

$P = 2xyz$ passes all tests. To find f we can solve the three equations, or notice that $f = xyz^2$ is successful. Its gradient is **F**.

A third method defines $f(x,y,z)$ as ***the work to reach*** (x,y,z) ***from*** $(0,0,0)$. The path doesn't matter. For practice we integrate $\mathbf{F} \cdot d\mathbf{R} = M\,dx + N\,dy + P\,dz$ along the straight line (xt, yt, zt):

$$f(x,y,z) = \int_0^1 (yt)(zt)^2(x\,dt) + (xt)(zt)^2(y\,dt) + 2(xt)(yt)(zt)(z\,dt) = xyz^2.$$

EXAMPLE 9 Why is div curl grad f automatically zero (in two ways)?

Solution First, curl grad f is zero (always). Second, div curl **F** is zero (always). Those are the key identities of vector calculus. We end with a review.

$$\textit{Green's Theorem}: \quad \oint \mathbf{F} \cdot d\mathbf{R} = \iint (\partial N/\partial x - \partial M/\partial y)\,dx\,dy$$

$$\oint \mathbf{F} \cdot \mathbf{n}\,ds = \iint (\partial M/\partial x + \partial N/\partial y)\,dx\,dy$$

$$\textit{Divergence Theorem}: \quad \iint \mathbf{F} \cdot \mathbf{n}\,dS = \iiint (\partial M/\partial x + \partial N/\partial y + \partial P/\partial z)\,dx\,dy\,dz$$

$$\textit{Stokes' Theorem}: \quad \oint \mathbf{F} \cdot d\mathbf{R} = \iint \text{curl}\,\mathbf{F} \cdot \mathbf{n}\,dS.$$

The first form of Green's Theorem leads to Stokes' Theorem. The second form becomes the Divergence Theorem. You may ask, ***why not go to three dimensions in the first place***? The last two theorems contain the first two (take $P = 0$ and a flat surface). We could have reduced this chapter to two theorems, not four. I admit that, but a fundamental principle is involved: "It is easier to generalize than to specialize."

For the same reason df/dx came before partial derivatives and the gradient.

15.6 EXERCISES

Read-through questions

The curl of $M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$ is the vector __a__. It equals the 3 by 3 determinant __b__. The curl of $x^2\mathbf{i}+z^2\mathbf{k}$ is __c__. For $\mathbf{S}=y\mathbf{i}-(x+z)\mathbf{j}+y\mathbf{k}$ the curl is __d__. This $\mathbf{S}$ is a __e__ field $\mathbf{a}\times\mathbf{R}=\frac{1}{2}(\text{curl }\mathbf{F})\times\mathbf{R}$, with axis vector $\mathbf{a}=$ __f__. For any gradient field $f_x\mathbf{i}+f_y\mathbf{j}+f_z\mathbf{k}$ the curl is __g__. That is the important identity curl grad $f=$ __h__. It is based on $f_{xy}=f_{yx}$ and __i__ and __j__. The twin identity is __k__.

The curl measures the __l__ of a vector field. A paddlewheel in the field with its axis along $\mathbf{n}$ has turning speed __m__. The spin is greatest when $\mathbf{n}$ is in the direction of __n__. Then the angular velocity is __o__.

Stokes' Theorem is __p__ = __q__. The curve C is the __r__ of the __s__ S. This is __t__ Theorem extended to __u__ dimensions. Both sides are zero when $\mathbf{F}$ is a gradient field because __v__.

The four properties of a conservative field are $\mathbf{A}=$ __w__, $\mathbf{B}=$ __x__, $\mathbf{C}=$ __y__, $\mathbf{D}=$ __z__. The field $y^2z^2\mathbf{i}+2xy^2z\mathbf{k}$ (passes)(fails) test $\mathbf{D}$. This field is the gradient of $f=$ __A__. The work $\int\mathbf{F}\cdot d\mathbf{R}$ from (0,0,0) to (1,1,1) is __B__ (on which path?). For every field $\mathbf{F}$, $\iint$ curl $\mathbf{F}\cdot\mathbf{n}ds$ is the same out through a pyramid and up through its base because __C__.

In Problems 1–6 find curl F.

1 $\mathbf{F}=z\mathbf{i}+x\mathbf{j}+y\mathbf{k}$

2 $\mathbf{F}=\text{grad}\,(xe^y\sin z)$

3 $\mathbf{F}=(x+y+z)(\mathbf{i}+\mathbf{j}+\mathbf{k})$

4 $\mathbf{F}=(x+y)\mathbf{i}-(x+y)\mathbf{k}$

5 $\mathbf{F}=\rho^n(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$

6 $\mathbf{F}=(\mathbf{i}+\mathbf{j})\times\mathbf{R}$

7 Find a potential f for the field in Problem 3.

8 Find a potential f for the field in Problem 5.

9 When do the fields $x^m\mathbf{i}$ and $x^n\mathbf{j}$ have zero curl?

10 When does $(a_1x+a_2y+a_3z)\mathbf{k}$ have zero curl?

In 11–14, compute curl F and find $\oint_C\mathbf{F}\cdot d\mathbf{R}$ by Stokes' Theorem.

11 $\mathbf{F}=x^2\mathbf{i}+y^2\mathbf{k}$, $C=$ circle $x^2+z^2=1, y=0$.

12 $\mathbf{F}=\mathbf{i}\times\mathbf{R}$, $C=$ circle $x^2+z^2=1, y=0$.

13 $\mathbf{F}=(\mathbf{i}+\mathbf{j})\times\mathbf{R}$, $C=$ circle $y^2+z^2=1, x=0$.

14 $\mathbf{F}=(y\mathbf{i}-x\mathbf{j})\times(x\mathbf{i}+y\mathbf{j})$, $C=$ circle $x^2+y^2=1, z=0$.

15 (important) Suppose two surfaces S and T have the same boundary C, and the direction around C is the same.

(a) Prove $\iint_S$ curl $\mathbf{F}\cdot\mathbf{n}dS=\iint_T$ curl $\mathbf{F}\cdot\mathbf{n}dS$.

(b) Second proof: The difference between those integrals is $\iiint$ div(curl $\mathbf{F}$)dV. By what Theorem? What region is V? Why is this integral zero?

16 In 15, suppose S is the top half of the earth ($\mathbf{n}$ goes out) and T is the bottom half ($\mathbf{n}$ comes in). What are C and V? Show by example that $\iint_S\mathbf{F}\cdot\mathbf{n}dS=\iint_T\mathbf{F}\cdot\mathbf{n}dS$ is *not* generally true.

17 Explain why $\iint$ curl $\mathbf{F}\cdot\mathbf{n}dS=0$ over the closed boundary of any solid V.

18 Suppose curl $\mathbf{F}=\mathbf{0}$ and div $\mathbf{F}=0$. (a) Why is $\mathbf{F}$ the gradient of a potential? (b) Why does the potential satisfy Laplace's equation $f_{xx}+f_{yy}+f_{zz}=0$?

In 19–22, find a potential f if it exists.

19 $\mathbf{F}=z\mathbf{i}+\mathbf{j}+x\mathbf{k}$

20 $\mathbf{F}=2xyz\mathbf{i}+x^2z\mathbf{j}+x^2y\mathbf{k}$

21 $\mathbf{F}=e^{x-z}\mathbf{i}-e^{x-z}\mathbf{k}$

22 $\mathbf{F}=yz\mathbf{i}+xz\mathbf{j}+(xy+z^2)\mathbf{k}$

23 Find a field with curl $\mathbf{F}=(1,0,0)$.

24 Find all fields with curl $\mathbf{F}=(1,0,0)$.

25 $\mathbf{S}=\mathbf{a}\times\mathbf{R}$ is a spin field. Compute $\mathbf{F}=\mathbf{b}\times\mathbf{S}$ (constant vector $\mathbf{b}$) and find its curl.

26 How fast is a paddlewheel turned by the field $\mathbf{F}=y\mathbf{i}-x\mathbf{k}$ (a) if its axis direction is $\mathbf{n}=\mathbf{j}$? (b) if its axis is lined up with curl $\mathbf{F}$? (c) if its axis is perpendicular to curl $\mathbf{F}$?

27 How is curl $\mathbf{F}$ related to the angular velocity ω in the spin field $\mathbf{F}=\omega(-y\mathbf{i}+x\mathbf{j})$? How fast does a wheel spin, if it is in the plane $x+y+z=1$?

28 Find a vector field $\mathbf{F}$ whose curl is $\mathbf{S}=y\mathbf{i}-x\mathbf{j}$.

29 Find a vector field $\mathbf{F}$ whose curl is $\mathbf{S}=\mathbf{a}\times\mathbf{R}$.

30 ***True or false:*** when two vector fields have the same curl at all points: (a) their difference is a constant field (b) their difference is a gradient field (c) they have the same divergence.

In 31–34, compute $\iint$ curl $\mathbf{F}\cdot\mathbf{n}dS$ over the top half of the sphere $x^2+y^2+z^2=1$ and (separately) $\oint\mathbf{F}\cdot d\mathbf{R}$ around the equator.

31 $\mathbf{F}=y\mathbf{i}-x\mathbf{j}$

32 $\mathbf{F}=\mathbf{R}/\rho^2$

33 $\mathbf{F}=\mathbf{a}\times\mathbf{R}$

34 $\mathbf{F}=(\mathbf{a}\times\mathbf{R})\times\mathbf{R}$

35 The circle C in the plane $x+y+z=6$ has radius r and center at (1,2,3). The field $\mathbf{F}$ is $3z\mathbf{j}+2y\mathbf{k}$. Compute $\oint\mathbf{F}\cdot d\mathbf{R}$ around C.

36 S is the top half of the unit sphere and $\mathbf{F}=z\mathbf{i}+x\mathbf{j}+xyz\mathbf{k}$. Find $\iint$ curl $\mathbf{F}\cdot\mathbf{n}dS$.

37 Find $g(x,y)$ so that curl $g\mathbf{k}=y\mathbf{i}+x^2\mathbf{j}$. What is the name for g in Section 15.3? It exists because $y\mathbf{i}+x^2\mathbf{j}$ has zero ____.

38 Construct $\mathbf{F}$ so that curl $\mathbf{F}=2x\mathbf{i}+3y\mathbf{j}-5z\mathbf{k}$ (which has zero divergence).

39 Split the field $\mathbf{F}=xy\mathbf{i}$ into $\mathbf{V}+\mathbf{W}$ with curl $\mathbf{V}=0$ and div $\mathbf{W}=0$.

40 Ampère's law for a steady magnetic field $\mathbf{B}$ is curl $\mathbf{B} = \mu\mathbf{J}$ ($\mathbf{J}$ = current density, μ = constant). Find the work done by $\mathbf{B}$ around a space curve C from the current passing through it.

Maxwell allows varying currents which brings in the electric field.

41 For $\mathbf{F} = (x^2 + y^2)\mathbf{i}$, compute curl (curl $\mathbf{F}$) and grad (div $\mathbf{F}$) and $\mathbf{F}_{xx} + \mathbf{F}_{yy} + \mathbf{F}_{zz}$.

42 For $\mathbf{F} = v(x, y, z)\mathbf{i}$, prove these useful identities:

(a) curl(curl $\mathbf{F}$) = grad (div $\mathbf{F}$) $- (\mathbf{F}_{xx} + \mathbf{F}_{yy} + \mathbf{F}_{zz})$.

(b) curl($f\mathbf{F}$) = f curl $\mathbf{F}$ + (grad f) $\times \mathbf{F}$.

43 If $\mathbf{B} = \mathbf{a}\cos t$(constant direction $\mathbf{a}$), find curl $\mathbf{E}$ from Faraday's Law. Then find the alternating spin field $\mathbf{E}$.

44 With $\mathbf{G}(x, y, z) = m\mathbf{i} + n\mathbf{j} + p\mathbf{k}$, write out $\mathbf{F} \times \mathbf{G}$ and take its divergence. Match the answer with $\mathbf{G} \cdot$ curl $\mathbf{F} - \mathbf{F} \cdot$ curl $\mathbf{G}$.

45 Write down Green's Theorem in the xz plane from Stokes' Theorem.

46 ***True or false***: $\nabla \times \mathbf{F}$ is perpendicular to $\mathbf{F}$.

47 (a) The second proof of Stokes' Theorem took $M^* = M(x, y, f(x, y)) + P(x, y, f(x, y))\partial f/\partial x$ as the M in Green's Theorem. Compute $\partial M^*/\partial y$ from the chain rule and product rule (there are five terms).

(b) Similarly $N^* = N(x, y, f) + P(x, y, f)\partial f/\partial y$ has the x derivative $N_x + N_z f_x + P_x f_y + P_z f_x f_y + P f_{yx}$. Check that $N_x^* - M_y^*$ matches the right side of equation (8), as needed in the proof.

48 "The shadow of the boundary is the boundary of the shadow." This fact was used in the second proof of Stokes' Theorem, going to Green's Theorem on the shadow. Give two examples of S and C and their shadows.

49 Which integrals are equal when C = boundary of S or S = boundary of V?

$\oint \mathbf{F} \cdot d\mathbf{R}$ $\quad \oint(\text{curl } \mathbf{F}) \cdot d\mathbf{R}$ $\quad \oint(\text{curl } \mathbf{F}) \cdot \mathbf{n}\, ds$ $\quad \iint \mathbf{F} \cdot \mathbf{n}\, dS$

$\iint \text{div } \mathbf{F}\, dS$ $\quad \iint(\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$ $\quad \iint(\text{grad div } \mathbf{F}) \cdot \mathbf{n}\, dS$ $\quad \iiint \text{div } \mathbf{F}\, dV$

50 Draw the field $\mathbf{V} = -x\mathbf{k}$ spinning a wheel in the xz plane. What wheels would *not* spin?