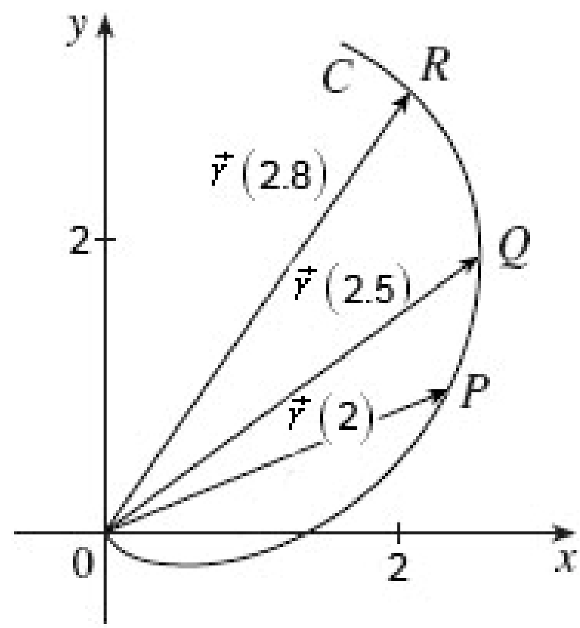
Math 2A – Vector Calculus – Fall ’11 – Chapter 13 Problems Name____________________
Show your work for credit. Do not use a calculator. Write all responses on separate paper.

1. Consider the curve $\vec{r}(t) = \langle x(t), y(t) \rangle = \langle 1+2\cos t, 2-\sin t \rangle$ with x and y in meters and t in secs.
   a. Sketch a graph for the curve showing the direction of motion.
   b. Simplify expressions for the velocity function $\vec{v}(t)$ (vector valued) and the speed (a scalar.)
   c. Find the curvature function and determine its maximum and minimum values.
   d. Find equations for the osculating circles where the curvature is maximum.
   e. Simplify the vector $\vec{h} = \vec{r} \times \vec{v}$. Is this a constant vector?
   f. Express the arclenth of the curve traced from $t = 0$ as an integral function, $s(t)$.

2. Find parametric equations for the curve at the intersection of $4x^2 + y^2 = 1$ and $x = z - 1$.

3. The figure shows the curve C of the vector function $\vec{r}(t)$.
   a. Draw on the diagram, $\vec{r}(2.8) - \vec{r}(2)$ and $\vec{r}(2.5) - \vec{r}(2)$. Label these.
   b. Draw on the diagram, $\dfrac{\vec{r}(2.8) - \vec{r}(2)}{0.8}$ and $\dfrac{\vec{r}(2.5) - \vec{r}(2)}{0.5}$. Label these.
   c. Draw the unit tangent vector, $\hat{T}(2)$



4. Consider the curves $\vec{r}_1(t) = \langle t, t+2, t^2+3 \rangle$ and $\vec{r}_2(u) = \langle u-3, u-1, u^3+6 \rangle$.
   a. Where do the curves intersect? Hint: This system is over-determined, so it’s sufficient to require their x and y coordinates are the same…then verify that the z coordinates also agree.
   b. Simplify an expression for the angle of intersection in terms of an arcos() function. This is the angle between the vectors tangent to the curves at that point.

5. Find an equation for the osculating plane of the curve $\vec{r}(t) = \langle 2t, 3t, t^3 + 3t \rangle$ at the point (2, 3, 4). Recall $\hat{B} = \hat{T} \times \hat{N}$ is normal to the osculating plane and that the equation for a plane is found by require that the dot product of a vector in the plane with the normal is zero.

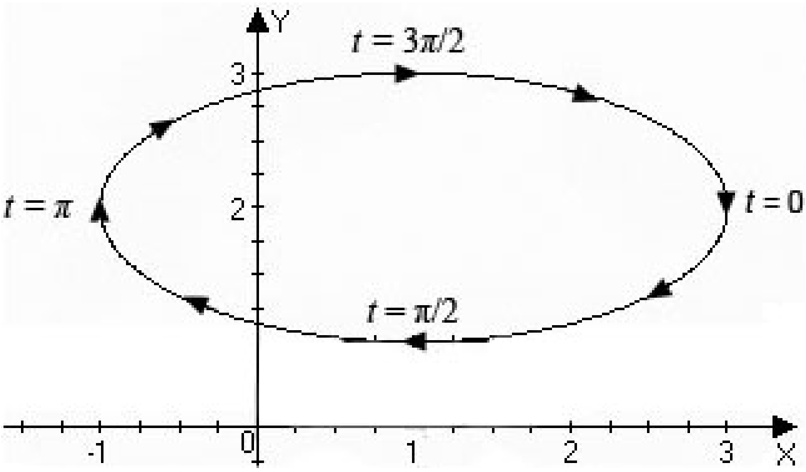
6. A particle starts at (0,0,100) and has initial velocity $\langle 1,2,3 \rangle$ and moves with acceleration $\vec{a}(t) = \langle -0.01, 0, -8 \rangle$. Find its position function and determine where it intersects the xy plane.

Math 2A – Vector Calculus – Fall '11 – Chapter 13 Test Solutions.

1. Consider the curve $\vec{r}(t) = \langle x(t), y(t)\rangle = \langle 1+2\cos t, 2-\sin t\rangle$ with x and y in meters and t in secs.

   a. Sketch a graph for the curve showing the direction of motion.

   SOLN: Here $\cos^2 t + \sin^2 t = \left(\frac{x-1}{2}\right)^2 + (y-2)^2 = 1$ so the curve is an ellipse centered at (1,2) with major axis from (–1,2) to (3,2) and minor axis from (1,1) to (1,3). The ellipse is traversed in the clockwise direction as shown:



   b. Simplify expressions for the velocity function $\vec{v}(t)$ (vector valued) and the speed (a scalar.)

   SOLN: The velocity is $\vec{v}(t) = \vec{r}\,'(t) = \langle x'(t), y'(t)\rangle = \langle -2\sin t, -\cos t\rangle$ so the speed is

   $|\vec{v}(t)| = \sqrt{4\sin^2 t + \cos^2 t} = \sqrt{1+3\sin^2 t}$

   c. Find the curvature function and determine its maximum and minimum values.

   SOLN: For simplicity and without loss of generalization, shift the ellipse so that it's centered at the origin with rectangular equation

$$x^2 + 4y^2 = 4.$$

   Differentiating with respect to x then yields

$$2x + 8yy' = 0 \text{ or } y' = -x/(4y).$$

   So

$$(ds/dx)^2 = 1 + x^2/(16y^2) = (16y^2 + x^{2)}/(16y^2)$$

   and

$$y'' = (-4y + 4xy')/(16y^2) = -(4y^2 + x^2)/y^3 = -4/y^3$$

   So

$$k(x) = |\, y'' \,| \,/\, (ds/dt)^3 = 256/(16 - 3x^2)^{3/2}.$$

   We could differentiate and set the derivative to zero, but a quick inspection shows that the denominator is smallest when $x = 0$ and since $x \leq 2$ we can conclude that the min occurs where $x = 0$ and the max occurs where $x = 2$. A good exercise is to repeat this derivation for x in terms of y. In terms of the original parametric curve, we see that the max curvature is where $t = 0$ or π and the min is where $t = \pi/2$ or $3\pi/2$.

You can also plug into this curvature formula,

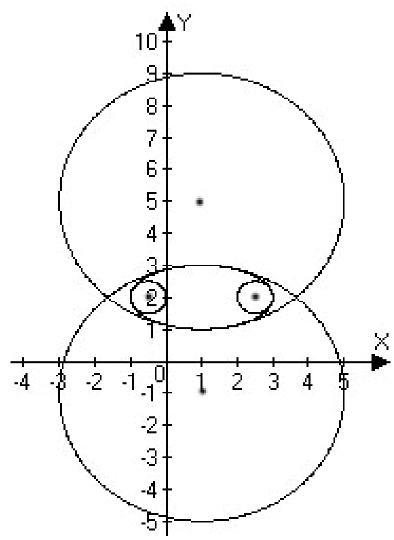
$$\kappa(t)=\frac{|\vec{v}(t)\times\vec{a}(t)|}{|\vec{v}(t)|^3}=\frac{|\langle -2\sin t,-\cos t,0\rangle\times\langle -2\cos t,\sin t,0\rangle|}{(1+3\sin^3 t)^{3/2}}=\frac{|-2\sin^2 t-2\cos^2 t|}{(1+3\sin^3 t)^{3/2}}=\frac{2}{(1+3\sin^3 t)^{3/2}}$$

By inspection of the ellipse it seems apparent that the max curvature is at $t = 0$ and $t = \pi$.

$$\kappa'(t)=\frac{d}{dt}\left(\frac{2}{(1+3\sin^3 t)^{3/2}}\right)=\frac{-\frac{3}{2}(9\sin^2 t)\cos t}{(1+3\sin^3 t)^{5/2}}=0\Leftrightarrow t=\frac{k\pi}{2}$$ where k is any integer.

This confirms the intuitive feeling of where the curvature of an ellipse is largest and where it's smallest. So the maximum curvature is $\kappa(0) = 2$ and the minimum is $\kappa(\pi/2) = 1/4$.

d. Find equations for the osculating circles where the curvature is maximum.
SOLN The radius of the osculating circle is the reciprocal of the curvature, thus the radius of the osculating circles where the curvature is maximized is ½ and the equations are $\left(x-\frac{5}{2}\right)^2+(y-2)^2=\frac{1}{4}$ and $\left(x+\frac{1}{2}\right)^2+(y-2)^2=\frac{1}{4}$.
At right is a graph of the ellipse with the four extreme osculating circles. The smaller circles have the greater curvature.



e. Simplify the vector $\vec{h}=\vec{r}\times\vec{v}$. Is this a constant vector?
SOLN:
$$\vec{r}(t)\times\vec{v}(t)=\langle 1+2\cos t,2-\sin t,0\rangle\times\langle -2\sin t,-\cos t,0\rangle$$
$$=\langle 0,0,-\cos t-2\cos^2 t+4\sin t-2\sin^2 t\rangle=\langle 0,0,-2-\cos t+4\sin t\rangle$$

Evidently this is not constant since at $t = 0$ its value is –3 and at $t = \pi$, it's –1.

f. Express the arclength of the curve traced from $t = 0$ as an integral function, $s(t)$.
SOLN: $s(t)=\int_0^t |r'(u)|\,du=\int_0^t |\langle -2\sin u,-\cos u\rangle|\,du=\int_0^t \sqrt{4\sin^2 u+\cos^2 u}\,du=\int_0^t \sqrt{3\sin^2 u+1}\,du$

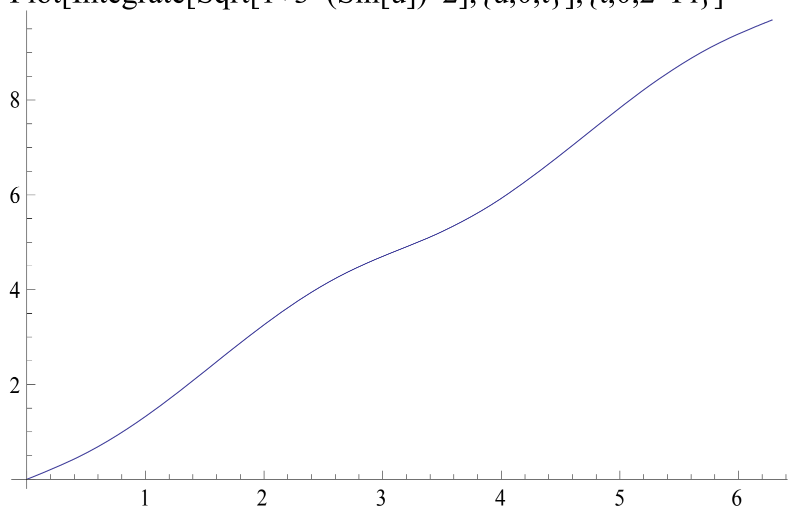
Entering "int(sqrt(1+3*(sin(u))^2),u,0,t)" into http://www.wolframalpha.com yields

$$\int_0^t \sqrt{1+3\sin^2(u)}\,du=$$
$$2\,E(-3)\,\text{IntegerPart}\left[\frac{t}{\pi}\right]+E\left(\pi\,\text{frac}\left(\frac{t}{\pi}\right)\middle|\,-3\right)\ \text{for } t\in\mathbb{R}$$

Entering “Integrate[Sqrt[1+3*(Sin[u])^2],{u,0,t}]” into Mathematica produces this: E(t|-3), and

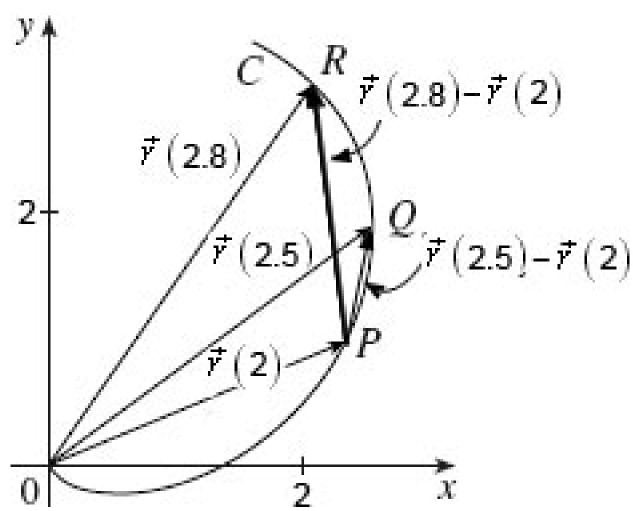
the plot command and graph are shown here:

Plot[Integrate[Sqrt[1+3*(Sin[u])^2],{u,0,t}],{t,0,2*Pi}]



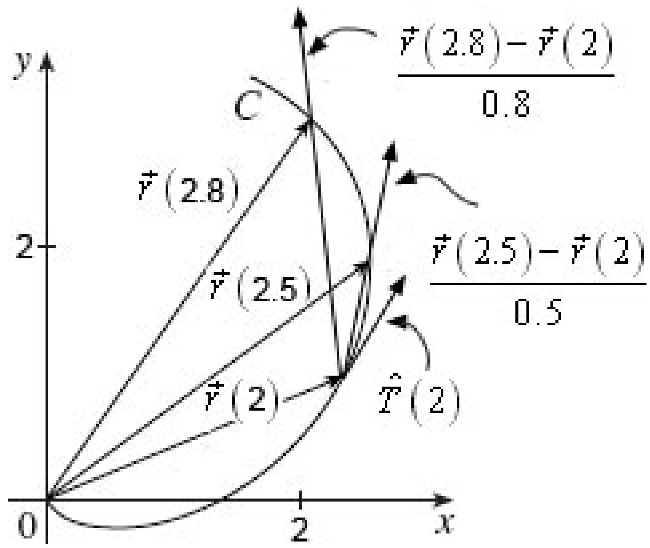
2. Find parametric equations for the curve at the intersection of $4x^2 + y^2 = 1$ and $x = z - 1$.
   SOLN: An ellipse in the *xy*-plane is projected onto the plane $z = x + 1$:
   $\vec{r}(t) = \left\langle \frac{1}{2}\cos t, \sin t, 1 + \frac{1}{2}\cos t \right\rangle$

3. The figure shows the curve C of the vector function $\vec{r}(t)$.
   a. Draw on the diagram, $\vec{r}(2.8) - \vec{r}(2)$ and $\vec{r}(2.5) - \vec{r}(2)$. Label these
      SOLN: See the diagram at right.



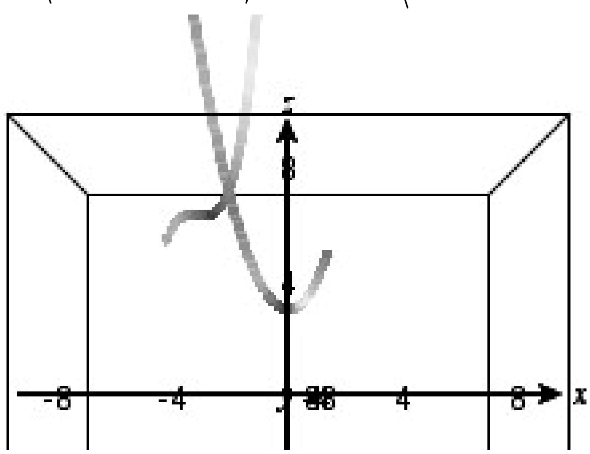
b. Draw on the diagram, $\dfrac{\vec{r}(2.8)-\vec{r}(2)}{0.8}$ and $\dfrac{\vec{r}(2.5)-\vec{r}(2)}{0.5}$. Label these.
SOLN: See the diagram at right →

c. Draw the unit tangent vector, $\hat{T}(2)$
SOLN: See the diagram at right →
Note that
$\lim\limits_{h\to 0}\dfrac{\vec{r}(2+h)-\vec{r}(2)}{h}=\vec{r}\,'(2)$ is not necessarily the unit tangent vector.



4. Consider the curves $\vec{r}_1(t)=\langle t,t+2,t^2+3\rangle$ and $\vec{r}_2(u)=\langle u-3,u-1,u^3+6\rangle$.

a. Where do the curves intersect? Hint: This system is over-determined, so it's sufficient to require their x and y coordinates are the same…then verify that the z coordinates also agree.
SOLN: Equating the x-coordinates we have $t = u – 3$, substituting this into the equation for the y-coordinates we have $(u – 3) + 2 = u – 1$, we have an identity. Plugging into the equation for the z-coordinates, $(u – 3)^2 + 3 = u^3 + 6$. Expanding and writing in equivalent descending powers form, $u^3 – u^2 + 6u – 6 = u^2(u – 1) + 6(u – 1) = (u^2 + 6)(u – 1) = 0$. So $u = 1$. How's that for simple? Thus $t = –2$ and the point of intersection is
$\vec{r}_2(1)=\langle 1-3,1-1,1^3+6\rangle=\vec{r}_1(-2)=\langle -2,-2+2,(-2)^2+3\rangle=\langle -2,0,7\rangle$



b. Simplify an expression for the angle of intersection in terms of an arcos() function. This is the angle between the vectors tangent to the curves at that point.
SOLN: First find the tangent vectors: $\vec{r}_2\,'(1)=\langle 1,1,3(1)^2\rangle=\langle 1,1,3\rangle$
$\vec{r}_1\,'(-2)=\langle 1,1,2(-2)\rangle\approx\langle 1,1,-4\rangle$ and so the angle between the curves is

$$\theta=\arccos(\frac{\vec{r}_2\,'(1)\bullet\vec{r}_1\,'(-2)}{|\vec{r}_2\,'(1)||\vec{r}_1\,'(-2)|}\approx\arccos\frac{\langle 1,1,3\rangle\bullet\langle 1,1,-4\rangle}{|\langle 1,1,3\rangle||\langle 1,1,-4\rangle|}\approx\arccos\frac{-10}{\sqrt{11}\sqrt{18}}\approx\arccos\left(-\frac{5\sqrt{22}}{33}\right)$$

Obtuse, no less!

5. Find an equation for the osculating plane of the curve $\vec{r}(t)=\langle 2t,3t,t^3+3t\rangle$ at the point (2, 3, 4). Recall $\hat{B}=\hat{T}\times\hat{N}$ is normal to the osculating plane and that the equation for a plane is found by require that the dot product of a vector in the plane with the normal is zero.
SOLN: $\vec{r}(1)=\langle 2,3,4\rangle$. The tangent (velocity) vector at t = 1, $\vec{r}\,'(1)=\langle 2,3,6\rangle$ is in the osculating plane, and the acceleration vector $\vec{r}\,''(1)=\langle 0,0,6\rangle$ also lies in the osculating plane. So compute a normal to the osculating plane as $\vec{r}\,'(1)\times\vec{r}\,''(1)=\langle 2,3,6\rangle\times\langle 0,0,6\rangle=\langle 18,-12,0\rangle=6\langle 3,-2,0\rangle$.
Thus an equation for the plane is $3(x-2)-2(y-3)=0$ or $3x-2y=0$. In fact, this curve is contained in that plane since for all t, $3(2t)-2(3t)=3x-2y=0$, so the same is true at any point.

6. A particle starts at (0,0,100) and has initial velocity $\langle 1,2,3\rangle$ and moves with acceleration $\vec{a}(t)=\langle -0.01,0,-8\rangle$. Find its position function and determine where it intersects the xy plane.
SOLN: $\vec{v}(t)=\int_0^t\vec{a}(u)\,du=\int_0^t\langle -0.01,0,-8\rangle du=\langle -0.01t,0,-8t\rangle+\vec{v}(0)=\langle -0.01t+1,2,-8t+3\rangle$

So
$$\vec{r}(t)=\int_0^t\vec{v}(u)\,du=\int_0^t\langle -0.01u+1,2,-8u+3\rangle du=\langle -0.005t^2+t,2t,-4t^2+3t\rangle+\vec{r}(0)$$
$$=\langle -0.005t^2+t,2t,-4t^2+3t+100\rangle$$

will intersect the xy-plane when $-4t^2+3t+100=0\Leftrightarrow\left(t-\frac{3}{8}\right)^2=25+\frac{9}{64}\Leftrightarrow t=\frac{3\pm\sqrt{1609}}{8}$