Math 2A – Vector Calculus – Final Exam – fall '11   Name________________________________

Show your work for credit. Do not use a calculator. Write all responses on separate paper.

1. Jon's tether broke while riding the Tilt-a-Whirl at an amusement park and flies off in free fall with an initial velocity of $v(0) = \langle 10,10,9.8\rangle$ from an initial position $p(0) = \langle 8,0,14.7\rangle$ relative to a coordinate system located at the base of the Tilt-a-Whirl. Jon lands in a vat of cotton candy on the ground. Where is the vat of cotton candy relative to that coordinate system?

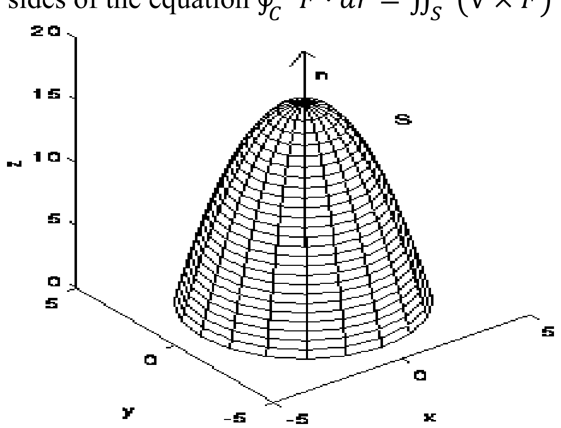
2. Consider the curve $\vec{r}(t) = \langle 2\sin \pi t\,, 2\cos \pi t\,, t\rangle$
   a. Find the unit tangent vector as a function of *t*.
   b. Find a normal vector as a function of *t*. Note: Doesn't have to be the unit normal.
   c. Find the curvature. *Hint*: The curvature is constant.
   d. Find an equation for the osculating plane where *t* = 1.
   e. Find the linear component a_T and normal component a_N of the acceleration so that $\vec{a}(t) = a_T\hat{T} + a_N\hat{N}$.

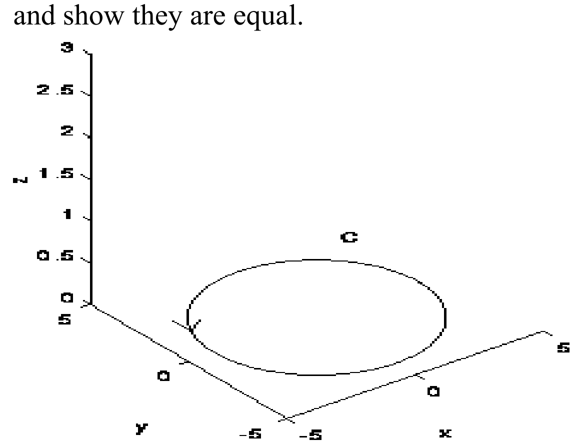
3. Consider the the ellipsoid $3x^2 + 2y^2 + z^2 = 9$
   a. Find an equation for the plane tangent to the ellipsoid at the point (1, 1, 2).
   b. Find the center of a sphere of radius 1 that has the same tangent plane at that point.

4. Let $w = -3x^2 - 4xy - y^2 - 12y + 16x$
   a. Find the critical points of *w* and classify each as either a max, a min, or a saddle point.
   b. Find the point in the first quadrant $x \geq 0,\ y \geq 0$ at which *w* is largest.

5. Find the flux of $\vec{F} = \langle x, y, -2z\rangle$ out of the surface *S* of the cube
   $C = \{(x,y,z)|0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$
   Hint: show that Gauss Theorem (the Divergence Theorem) applies here and use it.

6. Consider the surface described by the paraboloid $z = 16 - x^2 - y^2$ for $z \geq 0$, as shown below.
   Verify Stokes' Theorem for this surface and the vector field $\vec{F} = \langle 3y, 4z, -6x\rangle$. That is, evaluate both sides of the equation $\oint_C \vec{F}\cdot\overrightarrow{dr} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\overrightarrow{dS}$ and show they are equal.





7. Use the Divergence Theorem to compute the flux $\iint_S \vec{F}\cdot\overrightarrow{dS}$ where the surface *S* is the unit sphere $x^2 + y^2 + z^2 = 1$ and the vector field is $\vec{F} = \langle x^3, y^3, z^3\rangle$.

Math 2A – Vector Calculus – Final Exam Solutions – fall '11

1. Jon's tether broke while riding the Tilt-a-Whirl at an amusement park and flies off in free fall with an initial velocity of $v(0) = \langle 10,10,9.8\rangle$ from an initial position $p(0) = \langle 8,0,14.7\rangle$ relative to a coordinate system located at the base of the Tilt-a-Whirl. Jon lands in a vat of cotton candy on the ground. Where is the vat of cotton candy relative to that coordinate system?
   SOLN: The acceleration near the surface of Earth is $a = \langle 0,0,-9.8\rangle$.
   Thus Jon's velocity is $v(t) = \int_0^t \vec{a}(u)du + \vec{v}(0) = \langle 0,0,-9.8t\rangle + \langle 10,10,9.8\rangle = \langle 10,10,-9.8t+9.8\rangle$
   Jon's position at time *t* is then
   $p(t) = \int_0^t \vec{v}(u)du + \vec{p}(0) = \langle 10t,10t,-4.9t^2+9.8t\rangle + \langle 8,0,14.7\rangle = \langle 10t+8,10t,-4.9t^2+9.8t+14.7\rangle$
   So Jon will land in the vat when $-4.9t^2+9.8t+14.7 = -4.9(t^2-2t-3) = 0$, or when $t = 3$. Thus the vat is at $p(3) = \langle 38,30,0\rangle$.

2. Consider the curve $\vec{r}(t) = \langle 2\sin \pi t, 2\cos \pi t, t\rangle$
   a. Find the unit tangent vector as a function of *t*.
   SOLN: $\hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle 2\pi\cos\pi t,-2\pi\sin\pi t,1\rangle}{\sqrt{4\pi^2+1}}$
   b. Find a normal vector as a function of *t*. Note: Doesn't have to be the unit normal.
   SOLN: The rate of change of the unit vector is in the normal direction:
   $\frac{d}{dt}\hat{T}(t) = \frac{d}{dt}\frac{\langle 2\pi\cos\pi t,-2\pi\sin\pi t,1\rangle}{\sqrt{4\pi^2+1}} = \frac{-2\pi^2\langle \sin\pi t,\cos\pi t,0\rangle}{\sqrt{4\pi^2+1}}$, which is parallel to the unit vector $\hat{N} = -\langle \sin\pi t, \cos\pi t, 0\rangle$, pointing inwards from the position vector, not towards the origin, but directly towards the *z*-axis.
   c. Find the curvature. *Hint*: The curvature is constant.
   SOLN: Using the formula, $\kappa = \frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} = \frac{2\pi^2|\langle 2\pi\cos\pi t,-2\pi\sin\pi t,1\rangle\times\langle\sin\pi t,\cos\pi t,0\rangle|}{|4\pi^2+1|^{3/2}}$
   $$= \frac{2\pi^2|-\cos\pi t, \sin\pi t, 2\pi|}{|4\pi^2+1|^{3/2}} = \frac{2\pi^2}{4\pi^2+1}$$
   d. Find an equation for the osculating plane where $t = 1$.
   SOLN: $\vec{r}(1) = \langle 2\sin\pi, 2\cos\pi, 1\rangle = \langle 0,2,1\rangle$
   The osculating plane contains both the tangent and the normal vectors, so the cross product of those vectors is normal to the plane: $\vec{n} = \langle 2\pi\cos\pi t,-2\pi\sin\pi t,1\rangle \times \langle -\sin\pi t,-\cos\pi t,0\rangle = \langle \cos\pi t,-\sin\pi t,-2\pi\rangle$. At $t = 1$, a normal to the plane is thus $\langle 1,0,2\pi\rangle$
   An equation for the osculating plane is thus
   $$\langle 1,0,2\pi\rangle\cdot\langle x-0,y-2,z-1\rangle = x+2\pi(z-1) = 0$$
   e. Find the linear component a_T and normal component a_N of the acceleration so that $\vec{a}(t) = a_T\hat{T} + a_N\hat{N}$.
   SOLN: $\vec{a}(t) = \frac{d}{dt}\left(|\vec{r}\,'(t)|\hat{T}\right) = \hat{T}\frac{d}{dt}(|\vec{r}\,'(t)|) + |\vec{r}\,'(t)|\frac{d}{dt}(\hat{T}) = 0 + |\vec{r}\,'(t)|^2\kappa\hat{N} = 2\pi^2\hat{N}$.
   Since the speed is constant, $\vec{r}\,'(t) = \sqrt{4\pi^2+1}$, the linear acceleration is zero. The centripetal acceleration is $a_N = \kappa v^2 = 2\pi^2$

Here are some Mathematica graphics to illustrate what's going on here:

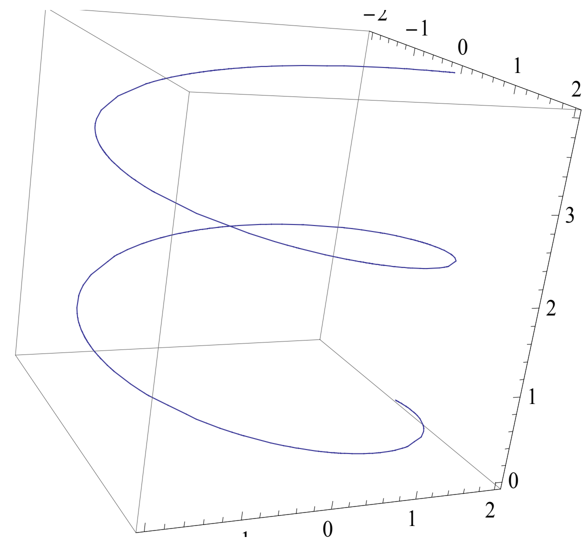
Helix = ParametricPlot3D[{2 * Sin[Pi * t], 2 * Cos[Pi * t], t}, {t, 0,4}]

Now, the radius of the osculating circle is

$$R = \frac{1}{\kappa} = \frac{1+4\pi^2}{2\pi^2},$$

so the center of the osculating circle is

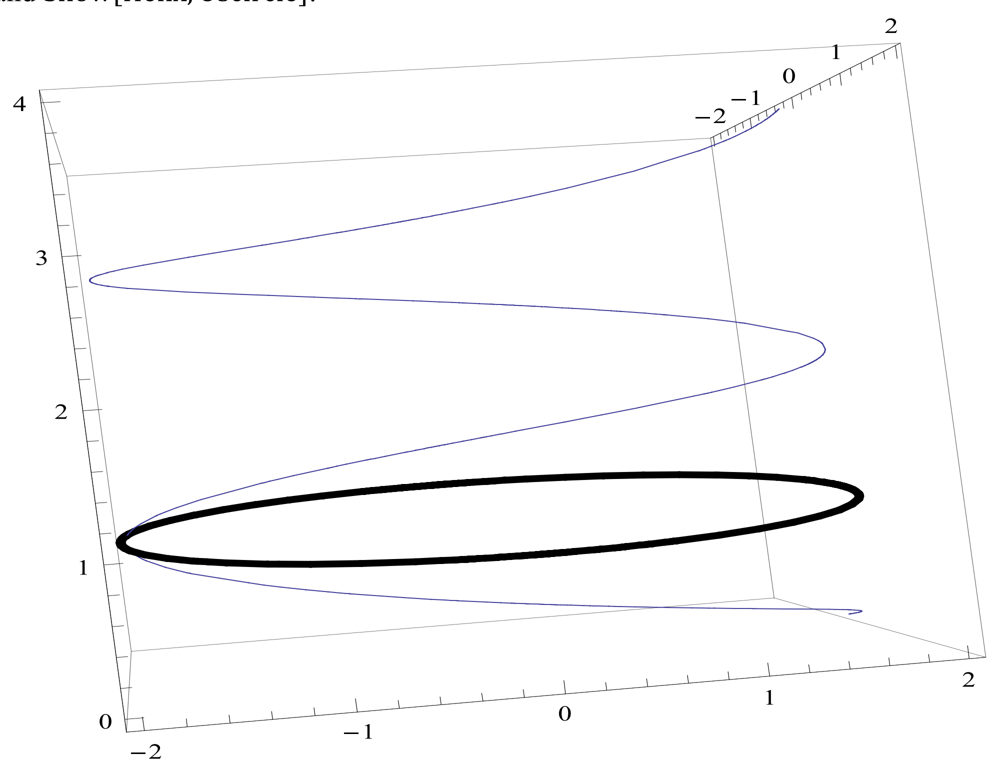
$$\vec{r}(1) + R\langle 0,1,0\rangle = \langle 0,-2,1\rangle + \langle 0, \frac{1+4\pi^2}{2\pi^2}, 0\rangle$$
$$= \langle 0, \frac{1}{2\pi^2}, 1\rangle.$$



The osculating circle is contained in the osculating plane , which is parallel to the y-axis, so the projection of the osculating circle on the xz-plane is a line segment centered at (0,0,1) of length $2R$ and having slope, $-\frac{1}{2\pi}$ .Thus the amplitude of oscillation for x is $\frac{2\pi}{\sqrt{4\pi^2+1}}$ and the amplitude of oscillation for z is $\frac{\sqrt{4\pi^2+1}}{4\pi^2}$. Thus we can graph the osculating circle together with the helix with

Oscircle = ParametricPlot3D[{−Sqrt[4 * Pi^2 + 1] * Cos[t]/Pi , 1/(4 * Pi^2) + (1 + 4 * Pi^2) * Sin[t]/(2 * Pi^2) , 1 + Sqrt[1 + 4 * Pi^2] * Cos[t]/(4 * Pi^2)}, {t, 0,2 * Pi}, PlotStyle → Thickness[0.01]]

and Show[Helix, Oscircle]:



Now it's up to you to figure out a formula for the evolute: the locus of centers of all osculating circles for this helix.

3. Consider the ellipsoid $3x^2 + 2y^2 + z^2 = 9$

a. Find an equation for the plane tangent to the ellipsoid at the point (1, 1, 2).

SOLN: We can parameterize the surface as either $\vec{r}(x,y) = \langle x, y, \sqrt{9-3x^2-2y^2}\rangle$ or as $\vec{r}(\theta,\phi) = \langle \theta, \phi, \frac{3}{\sqrt{\sin^2\phi(1+\cos^2\theta)+1}}\rangle$. The first of these looks a bit simpler to work with, so we compute the normal to the surface as the cross product of tangents:

$$\vec{n} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -\frac{3}{\sqrt{9-3x^2-2y^2}} \\ 0 & 1 & -\frac{2}{\sqrt{9-3x^2-2y^2}} \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -1 \end{vmatrix} = \langle \tfrac{3}{2}, 1, 1\rangle$$ and an equation for the tangent plane is obtained by using the fact that a vector parallel to the plane is perpendicular to the normal, making the dot product of these zero:

$\langle 1.5,1,1\rangle \cdot \langle x-1, y-1, z-2\rangle = 0$ or $z = \frac{9}{2} - \frac{3}{2}x - y$.

b. Find the center of a sphere of radius 1 that has the same tangent plane at that point.

SOLN: A unit vector in the direction of the normal is $\frac{\sqrt{17}}{17}\langle 3,2,2\rangle$ Just add (or subtract) the vectors:

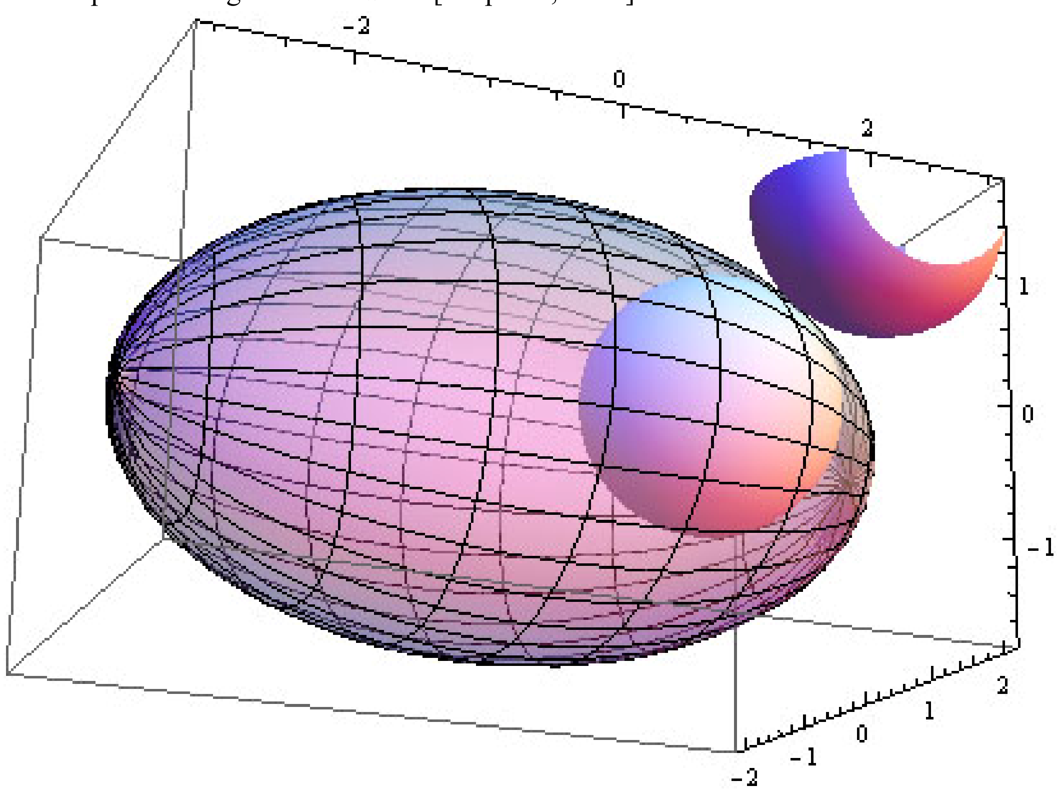
$\langle 1,1,2\rangle \pm \frac{\sqrt{17}}{17}\langle 3,2,2\rangle$

As an afterthought, In Mathematica, you can render the ellipsoid like so:

Ellipzoid =
SphericalPlot3D[3/Sqrt[(Sin[ϕ])^2 * (1 + (Cos[θ])^2) + 1] , {ϕ, 0,2Pi}, {θ, −Pi/2 , Pi/2}, PlotStyle → Opacity[0.5]]

And the two unit balls with: Balls = Graphics3D[{Sphere[{1 + 3/Sqrt[17] , 1 + 2/Sqrt[17] , 2 + 2/Sqrt[17]}, 1], Sphere[{1 − 3/Sqrt[17] , 1 − 2/Sqrt[17] , 2 − 2/Sqrt[17] }, 1]}]

and then put these together with Show[Ellipzoid, Balls]:



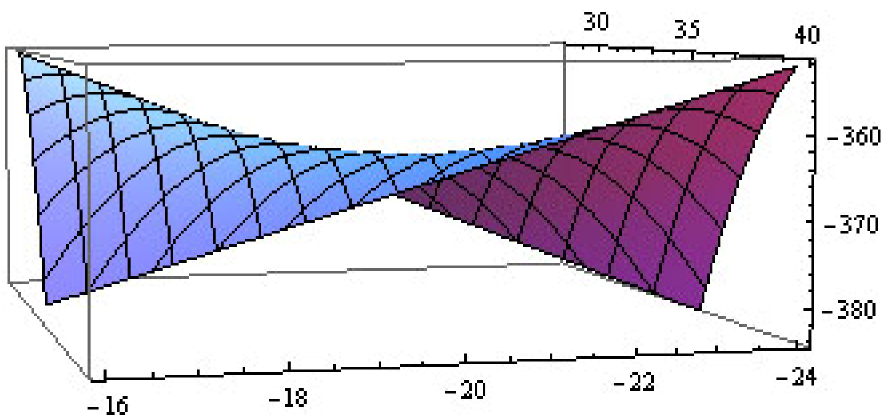
4. Let $w = -3x^2 - 4xy - y^2 - 12y + 16x$

a. Find the critical points of w and classify each as either a max, a min, or a saddle point.

SOLN: Critical points are where partial derivatives are both zero: $\frac{\partial}{\partial x} f(x,y) = -6x - 4y + 16 = 0$ and $\frac{\partial}{\partial y} f(x,y) = -4x - 2y - 12 = 0$. Solving the system yields only one critical point $(x,y) = (-20,34)$. Inspection of the function suggests this is clearly a maximum, but, to be sure, the second derivative test yields the discriminant $D = \begin{vmatrix} -6 & -4 \\ -4 & -2 \end{vmatrix} = -4 < 0$, which means that $w(-20,34) = 44$ is neither a maximum nor a minimum – it's a saddle.

The Mathematica command,

Plot3D$[-3 * x^2 - 4x * y - y^2 - 12 * y + 16 * x, \{x, -24, -16\}, \{y, 12 - x, 16 - x\}]$ illustrates this:



b. Find the point in the first quadrant $x \geq 0,\ y \geq 0$ at which w is largest.

SOLN: There are no critical points in this region, so the maximum will occur at the boundary: either along the *xw*-plane or along the *yw*-plane. Plugging in $x = 0$ we have $w = 36 - (y+6)^2$, giving a local max at $(0,-6,36)$. Plugging in $y = 0$ we have $w = \frac{64}{3} - 3\left(x - \frac{8}{3}\right)^2$ yielding a local max at $\left(\frac{8}{3}, 0, \frac{64}{3}\right)$. Since $36 > \frac{64}{3}$, the maximum value in the first quadrant is 36.

5. Find the flux of $\vec{F} = \langle x, y, -2z\rangle$ out of the surface S of the cube $C = \{(x,y,z) | 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$

Hint: show that Gauss Theorem (the Divergence Theorem) applies here and use it.

SOLN: The simplest way to work this is to use the divergence theorem and note that the divergence of $\vec{F}$ is $\vec{\nabla} \cdot \vec{F} = 1 + 1 - 2 = 0$ so the flux must be zero.

To evaluate the surface integrals directly means evaluating

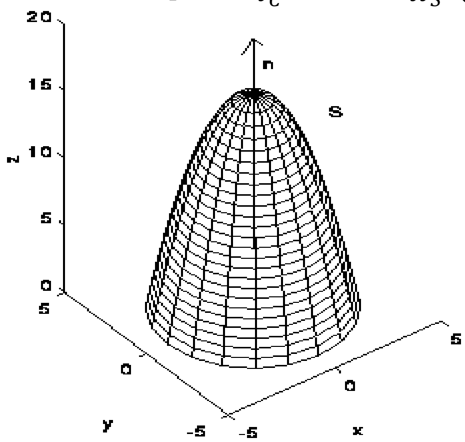
$$\oiint_S \vec{F} \cdot \overrightarrow{dS} = \iint_{x=0} \vec{F} \cdot \overrightarrow{dS} + \iint_{x=1} \vec{F} \cdot \overrightarrow{dS} + \iint_{y=0} \vec{F} \cdot \overrightarrow{dS} + \iint_{y=1} \vec{F} \cdot \overrightarrow{dS} + \iint_{z=0} \vec{F} \cdot \overrightarrow{dS} + \iint_{z=1} \vec{F} \cdot \overrightarrow{dS}$$

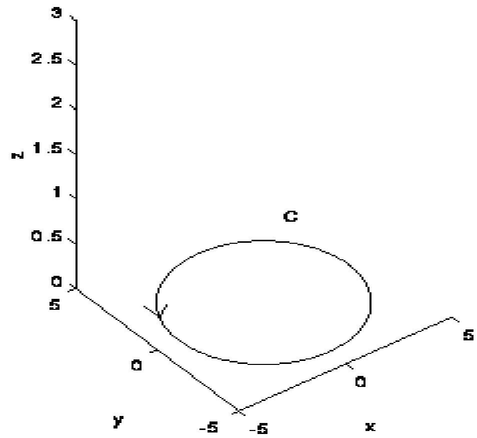
$$= \iint_{x=0} \langle 0, y, -2z\rangle \cdot \langle -1,0,0\rangle dA + \int_0^1 \int_0^1 \langle 1, y, z\rangle \cdot \langle 1,0,0\rangle dydz$$

$$+ \iint_{y=0} \langle x, 0, -2z\rangle \cdot \langle 0,-1,0\rangle dA + \int_0^1 \int_0^1 \langle x, 1, z\rangle \cdot \langle 1,0,0\rangle dxdz$$

$$+ \iint_{y=0} \langle x, y, 0\rangle \cdot \langle 0,0,-1\rangle dA + \int_0^1 \int_0^1 \langle x, y, -2\rangle \cdot \langle 0,0,1\rangle dxdy = 0 + 1 + 0 + 1 + 0 - 2 = 0$$

6. Consider the surface described by the paraboloid $z = 16 - x^2 - y^2$ for $z \geq 0$, as shown below.
Verify Stokes' Theorem for this surface and the vector field $\vec{F} = \langle 3y, 4z, -6x \rangle$. That is, evaluate both sides of the equation $\oint_C \vec{F} \cdot \overrightarrow{dr} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \overrightarrow{dS}$ and show they are equal.





SOLN: $\vec{r}(t) = \langle 4\cos t, 4\sin t\rangle \Rightarrow \vec{r}\,'(t) = \langle -4\sin t, 4\cos t\rangle$ so that

$\oint_C \vec{F} \cdot \overrightarrow{dr} = \int_0^{2\pi} \langle 12\sin t, 0, -24\cos t\rangle \cdot \langle -4\sin t, 4\cos t\rangle dt = -48\int_0^{2\pi} \sin^2 t\, dt = -48\pi$

Now the surface is parameterized by $\vec{r}(x,y) = \langle x, y, 16 - x^2 - y^2\rangle$ so a measure of the scaled infinitesimal surface normal is $\overrightarrow{dS} = (\vec{r}_x \times \vec{r}_y)dxdy = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} dxdy = \langle 2x, 2y, 1\rangle dxdy$ so that

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \overrightarrow{dS} = \iint_S \langle -4, 6, -3\rangle \cdot \overrightarrow{dS} = \int_{-4}^{4} \int_{-\sqrt{16-x}}^{\sqrt{16-x}} \langle -4, 6, -3\rangle \cdot \langle 2x, 2y, 1\rangle dxdy$$

$$= \int_0^4 \int_0^{2\pi} -8r^2\cos\theta + 12r^2\sin\theta - 3r d\theta dr = -6\pi \int_0^4 r dr = -48\pi$$

7. Use the Divergence Theorem to compute the flux $\iint_S \vec{F} \cdot \overrightarrow{dS}$ where the surface S is the unit sphere $x^2 + y^2 + z^2 = 1$ and the vector field is $\vec{F} = \langle x^3, y^3, z^3\rangle$.

SOLN: $\oiint_S \vec{F} \cdot \overrightarrow{dS} = \iiint_E \vec{\nabla} \cdot \vec{F} dV = \iiint_E 3x^2 + 3y^2 + 3z^2 dV = 3\int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^4 \sin\phi\, d\phi d\theta d\rho = -\frac{6\pi}{5}\cos\phi\big|_0^{\pi} = \frac{12\pi}{5}$