Math 2A – Vector Calculus – Fall '09 – Chapter 10 Test   Name________________

Show your work for credit. Write all responses on separate paper. Do not use a calculator.

1. Show that the parameterization of the intersection of the cylinder $x^2 + y^2 = 1$ with the parabolic sheet $z = xy$ can be parameterized by $\vec{r}(t) = \langle \cos t, \sin t, \sin 2t \rangle$.

2. Suppose the position (in meters) of a particle at time t (in seconds) is given by $\vec{r}(t) = \langle \cos 2t, \cos t, \sin t \rangle$
   a. Simplify an expression for the speed of the particle at time t.
   b. Find the distance the particle has traveled as t goes from 0 to π.
      Hint: to do the integral, it may be helpful to recall that $\sin^2\theta = \frac{1-\cos 2\theta}{2}$.

3. Consider the curve described by the vector function $\vec{r}(t) = \langle \cos t, 2\sin t \rangle$
   a. Make a large, careful sketch of the curve in the xy-plane.
   b. Draw the vectors $\vec{r}\left(\frac{\pi}{6}\right), \vec{r}\left(\frac{\pi}{3}\right)$ and $\vec{r}\left(\frac{\pi}{3}\right) - \vec{r}\left(\frac{\pi}{6}\right)$
   c. Draw the vector $\vec{r}\,'\left(\frac{\pi}{4}\right)$ and compare it with $\vec{r}\left(\frac{\pi}{3}\right) - \vec{r}\left(\frac{\pi}{6}\right)$

4. Recall that $\vec{r}\,' = \frac{ds}{dt}\hat{T}$.
   a. Use this derive that $\vec{r}\,' \times \vec{r}\,'' = \left(\frac{ds}{dt}\right)^2 \left(\hat{T} \times \hat{T}'\right)$
   b. Now use the formula $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta$ to show how to simplify $|\vec{r}\,' \times \vec{r}\,''|$ and deduce that $|\hat{T}'| = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^2}$
   c. Simplify $|\hat{T}'| = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^2}$ for $r(x) = \langle x, f(x), 0 \rangle$.

5. Consider the curve of the parabola given by $y = x - \frac{x^2}{4}$.
   Find an equation for the osculating circle where $x = 2$.

6. Consider the curve $\vec{r}(t) = \langle \cos t, \sqrt{2}\,t, \sin t \rangle$
   a. Find the velocity, acceleration and speed of particle as functions of t.
   b. Find the normal vector as a function of t.
   c. Find the normal and tangential components of the tangent vector when $t = 0$.

7. Paramaterize the part of the sphere $\rho = 8$ that lies between the planes $z = 4\sqrt{3}$ and $z = -4\sqrt{3}$
   a. Using cylindrical coordinates. Includes appropriate intervals for the parameters r and θ.
   b. Using spherical coordinates. Include appropriate intervals for the parameters θ and Φ.

8. Find parametric equations for the surface obtained by rotating $y = \sin x$ about the x-axis.ghag

Problem 3. A ladybug is climbing on a Volkswagen Bug (= VW). In its starting position, the surface of the VW is represented by the unit semicircle $x^2 + y^2 = 1$, $y \geq 0$ in the xy-plane. The road is represented as the x-axis. At time $t = 0$ the ladybug starts at the front bumper, (1, 0), and walks counterclockwise around the VW at unit speed relative to the VW. At the same time the VW moves to the right at speed 10.

a) Find the parametric formula for the trajectory of the ladybug, and find its position when it reaches the rear bumper. (At $t = 0$, the rear bumper is at $(-1, 0)$.)

b) Compute the speed of the bug, and find where it is largest and smallest. Hint: It is easier to work with the square of the speed.

Math 2A – Vector Calculus – Fall '09 – Chapter 10 Test Solutions

1. Show that the parameterization of the intersection of the cylinder $x^2 + y^2 = 1$ with the parabolic sheet $z = 2xy$ can be parameterized by $\vec{r}(t) = \langle \cos t, \sin t, \sin 2t \rangle$.

   SOLN: $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ follows directly from the Pythagorean identity.
   $2xy = 2\cos(t)\sin(t) = \sin(2t) = z$ is immediate from the double angle identity for sine.

2. Suppose the position (in meters) of a particle at time t (in seconds) is given by $\vec{r}(t) = \langle \cos 2t, \cos t, \sin t \rangle$ (note that this is, like #1, a "pringle" shape)

   a. Simplify an expression for the speed of the particle at time t.

   SOLN: $v(t) = |\vec{v}\,'| = \sqrt{(-2\sin 2u)^2 + (-\sin u)^2 + (\cos u)^2} = \sqrt{4\sin^2 2u + 1}$

   b. Find the distance the particle has traveled as t goes from 0 to π.

   SOLN: $D = \int_0^\pi \sqrt{4\sin^2 2u + 1}\, du = \int_0^\pi \sqrt{4\left(\frac{1-\cos 4u}{2}\right) + 1}\, du = \int_0^\pi \sqrt{3 - 2\cos 4u}\, du$

   Since there is no elementary antiderivative, this presents a bit of a problem. One solution is to introduce a new class of antiderivatives. This is what Mathematica does, for example.

   Mathematica defines $E(t\,|\,m) = \int_0^t \sqrt{1 - m\sin^2 u}\, du$, for $-\pi/2 < t < \pi/2$, so that, in Mathematica you enter

   Integrate[Sqrt[3 − 2Cos[4u]], u],

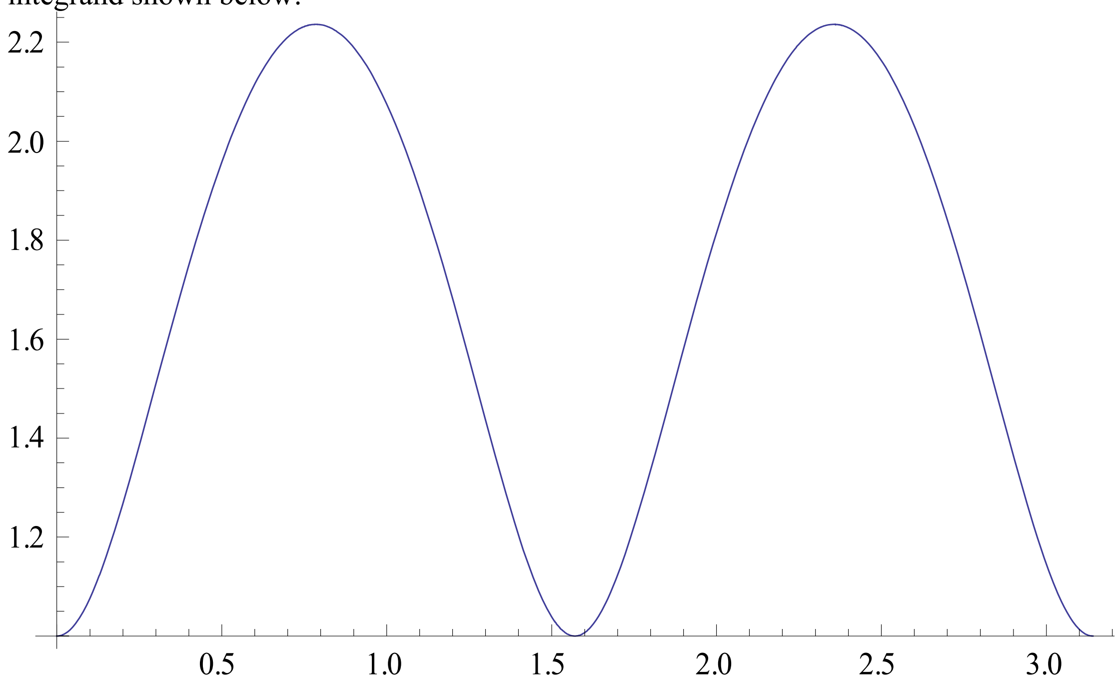
   it returns

   $\frac{1}{2}$EllipticE[2u, −4]

   Now, by symmetry, the integral from 0 to π is 4 times the integral from 0 to $\pi/4$. Thus,
   $D = 4\int_0^{\pi/4} \sqrt{1 + 4\sin^2 2u}\, du$ simplifies in Mathematica to

   EllipticE[−4],

   which when approximated yields about 5.27037, a result that comports with the graph of the integrand shown below:

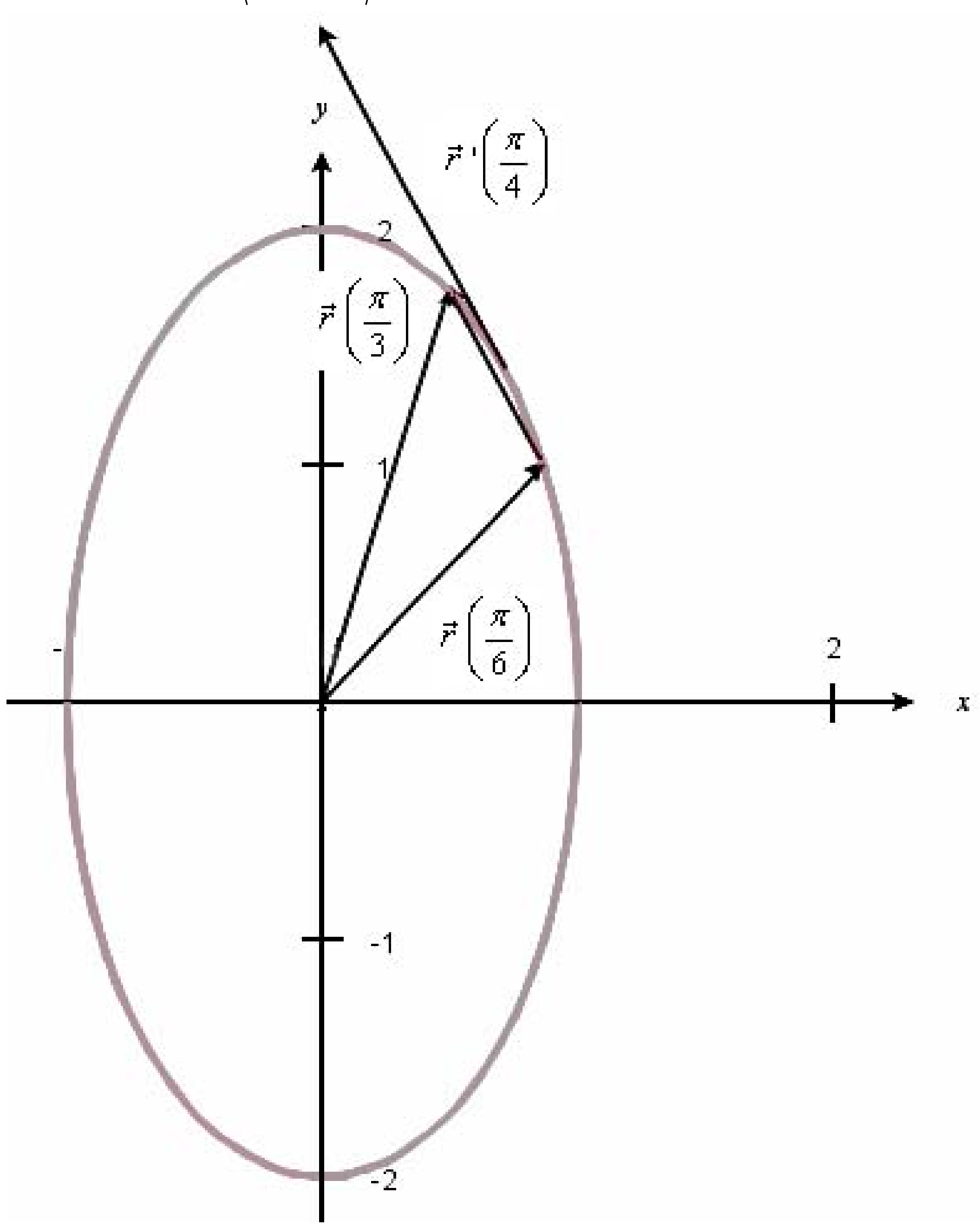


3. Consider the curve described by the vector function $\vec{r}(t) = \langle \cos t, 2\sin t \rangle$
   a. Make a large, careful sketch of the curve in the *xy*-plane.
   b. Draw the vectors $\vec{r}\left(\frac{\pi}{6}\right), \vec{r}\left(\frac{\pi}{3}\right)$ and $\vec{r}\left(\frac{\pi}{3}\right) - \vec{r}\left(\frac{\pi}{6}\right)$
   c. Draw the vector $\vec{r}\,'\left(\frac{\pi}{4}\right)$ and compare it with $\vec{r}\left(\frac{\pi}{3}\right) - \vec{r}\left(\frac{\pi}{6}\right)$

   SOLN: The idea here is that

$$\vec{r}\left(\frac{\pi}{3}\right) - \vec{r}\left(\frac{\pi}{6}\right) = \left\langle \frac{1}{2}, \sqrt{3} \right\rangle - \left\langle \frac{\sqrt{3}}{2}, 1 \right\rangle \approx \langle -0.366, 0.732 \rangle = \frac{\langle -0.732, 1.464 \rangle}{2} \approx \frac{\langle -0.707, 1.414 \rangle}{2}$$

$$\approx \frac{1}{2}\left\langle -\frac{\sqrt{2}}{2}, \sqrt{2} \right\rangle = \frac{\vec{r}\,'\left(\frac{\pi}{4}\right)}{2}$$



4. Recall that $\vec{r}\,' = \frac{ds}{dt}\hat{T}$.

   a. Use this derive that $\vec{r}\,'\times\vec{r}\,'' = \left(\frac{ds}{dt}\right)^2\left(\hat{T}\times\hat{T}\,'\right)$

   SOLN: By the product rule, $\vec{r}\,'' = \frac{ds}{dt}\left(\frac{ds}{dt}\hat{T}\right) = \frac{d^2s}{dt^2}\hat{T} + \frac{ds}{dt}\frac{d\hat{T}}{dt}$ so that

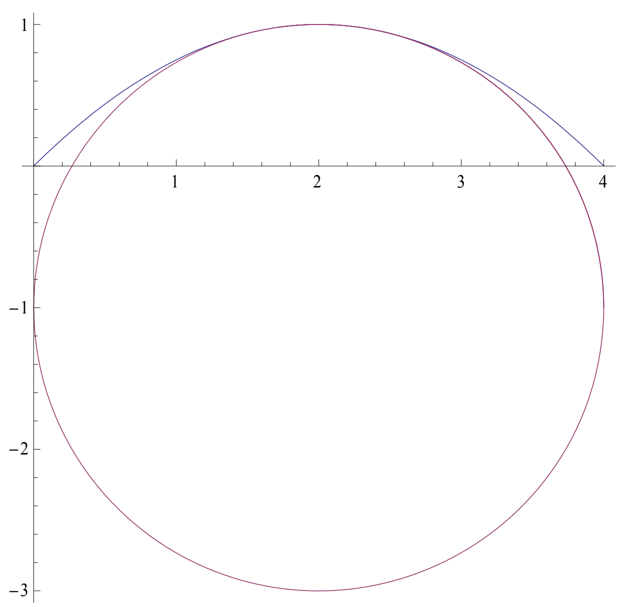
   $\vec{r}\,'\times\vec{r}\,'' = \frac{ds}{dt}\hat{T}\times\left(\frac{d^2s}{dt^2}\hat{T} + \frac{ds}{dt}\frac{d\hat{T}}{dt}\right) = 0 + \left(\frac{ds}{dt}\right)^2\left(\hat{T}\times\hat{T}\,'\right)$ where the 0 is the result of crossing two parallel vectors.

   b. Now use the formula $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$ to show how to simplify $|\vec{r}\,'\times\vec{r}\,''|$ and deduce that $\left|\hat{T}\,'\right| = \frac{|\vec{r}\,'\times\vec{r}\,''|}{|\vec{r}\,'|^2}$

   SOLN: $|\vec{r}\,'\times\vec{r}\,''| = \left(\frac{ds}{dt}\right)^2\left|\hat{T}\times\hat{T}\,'\right| = |\vec{r}\,'|^2\left|\hat{T}\right|\left|\hat{T}\,'\right| = |\vec{r}\,'|^2\left|\hat{T}\,'\right| \Leftrightarrow \left|\hat{T}\,'\right| = \frac{|\vec{r}\,'\times\vec{r}\,''|}{|\vec{r}\,'|^2}$ where the second equality is justified by observing that, since $\hat{T}$, has constant length, $\hat{T}\,'\perp\hat{T}$

   c. Simplify $\left|\hat{T}\,'\right| = \frac{|\vec{r}\,'\times\vec{r}\,''|}{|\vec{r}\,'|^2}$ for $r(x) = \langle x, f(x), 0\rangle$.

   SOLN: $\left|\hat{T}\,'\right| = \frac{\left|\langle 1, f\,'(x), 0\rangle\times\langle 0, f\,''(x), 0\rangle\right|}{\left|\sqrt{1+\left(f\,'(x)\right)^2}\right|^2} = \frac{|f\,''(x)|}{1+\left(f\,'(x)\right)^2}$



Consider the curve of the parabola given by $y = x - \frac{x^2}{4}$. Find an equation for the osculating circle where $x = 2$. SOLN: $\kappa = \frac{\left|\hat{T}\,'\right|}{|\vec{r}\,'|}$

$$= \left.\frac{|f\,''(x)|}{\left(1+\left(f\,'(x)\right)^2\right)^{3/2}}\right|_{x=2} = \left.\frac{1/2}{\left(1+\left(1-x/2\right)^2\right)^{3/2}}\right|_{x=2} = \frac{1}{2}$$

Thus the radius of the osculating circle is 2, and since $y = x - \frac{x^2}{4} = -\frac{1}{4}(x-2)^2 + 1$, the point, (2,1) is the vertex of the parabola and thus the center of the osculating circle is 2 units below (2,1) at (2, –1). Thus the equation of the osculating circle is $(x-2)^2 + (y+1)^2 = 4$.
See the diagram at right.

5. Consider the curve $\vec{r}(t) = \left\langle \cos t, \sqrt{2}\,t, \sin t\right\rangle$

a. Find the velocity, acceleration and speed of particle as functions of *t*.
SOLN: The velocity is $\vec{r}'(t) = \langle -\sin t, \sqrt{2}, \cos t \rangle$.
The acceleration is $\vec{r}''(t) = \langle -\cos t, 0, -\sin t \rangle$ and the speed is
$|\vec{r}'(t)| = \sqrt{(-\sin t)^2 + (\sqrt{2})^2 + (\cos t)^2} = \sqrt{3}$

b. Find the unit normal vector as a function of *t*.
SOLN: The unit tangent vector is $\hat{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle -\sin t, \sqrt{2}, \cos t \rangle}{\sqrt{3}}$, thus the unit normal vector is
$\hat{N} = \frac{\hat{T}'}{|\hat{T}|} = \langle -\cos t, 0, -\sin t \rangle$

c. Find the normal and tangential components of the tangent vector when *t* = 0.
SOLN: $a_T = \frac{d^2 s}{dt^2} = 0$ and, since the only component of the acceleration is in the normal (centripetal) direction $a_N = \kappa v^2 = |\vec{r}''| = 1$.

6. Paramaterize the part of the sphere $\rho = 8$ that lies between the planes $z = 4\sqrt{3}$ and $z = -4\sqrt{3}$
   a. Using cylindrical coordinates. Includes appropriate intervals for the parameters *r* and *θ*.
   SOLN: The equation for the sphere in cylindrical coordinates is $r^2 + z^2 = 8^2$ where $r = 8\sin\phi$.
   $z = 8\cos\phi = 4\sqrt{3} \Leftrightarrow \cos\phi = \frac{\sqrt{3}}{2}$, so the range of ϕ that gives the desired portion of the sphere is $0 \le \theta \le 2\pi$ and so *r* ranges between 4 and 8.
   b. Using spherical coordinates. Include appropriate intervals for the parameters *θ* and *Φ*.
   SOLN: $\frac{\pi}{6} \le \phi \le \frac{5\pi}{6}$ and $0 \le \theta \le 2\pi$

7. Find parametric equations for the surface obtained by rotating *y* = sin*x* about the *x*-axis.
SOLN: The circle of radius sin*x*, parallel to the *xz*-plane and centered at (*x*,0,0) can be parameterized by *y* = sin*x* cos*θ* and *z* = sin*x* sin*θ*. Thus the parameterization we seek is
$\langle x, y, z \rangle = \langle x, \sin x \cos\theta, \sin x \sin\theta \rangle$

Problem 3. A ladybug is climbing on a Volkswagen Bug (= VW). In its starting position, the surface of the VW is represented by the unit semicircle $x^2 + y^2 = 1$, $y \geq 0$ in the *xy*-plane. The road is represented as the *x*-axis. At time $t = 0$ the ladybug starts at the front bumper, (1, 0), and walks counterclockwise around the VW at unit speed relative to the VW. At the same time the VW moves to the right at speed 10.

a) Find the parametric formula for the trajectory of the ladybug, and find its position when it reaches the rear bumper. (At $t = 0$, the rear bumper is at $(-1, 0)$.)

b) Compute the speed of the bug, and find where it is largest and smallest. Hint: It is easier to work with the square of the speed.

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