Math 2A Final Exam Problems for Consideration

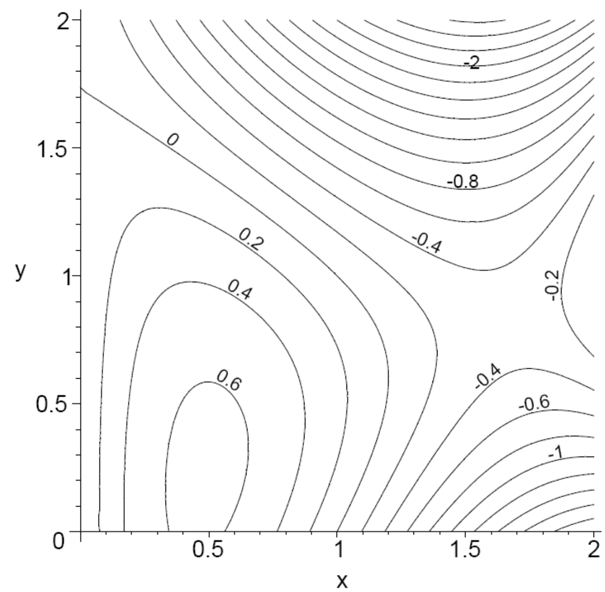
1. Show all work for credit. Be sure to show what you know.

   Given points $A(1,0,1)$, $B(2,3,0)$, $C(-1, 1, 4)$ and $D(0,3,2)$, find the volume of the parallelepiped with adjacent edges AB, AC, and AD. Hint: The volume of the parallelepiped determined by vectors $\vec{a},\vec{b},\vec{c}$ is the triple product, $\left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right|$.

2. Consider the lines $\vec{r}_1(t)=\langle 1+2t, 3+4t, 5+6t\rangle$ and $\vec{r}_2(t)=\langle 4t-5, 3t+1, 2t+7\rangle$.
   a. Show that these lines intersect by finding the point of intersection.
   b. Find an equation for the plane containing these lines.

3. Let $z=x^y\left(=e^{y\ln x}\right)$. Give an approximate formula for the small change Δz that results from small changes Δx and Δy in the values of x and y. Use this to approximate the value of $1.99^{3.02}$.

4. The figure at right is the contour plot of a function of two variables, $f(x,y)$, for x and y ranging from 0 to 2. The scale is 1 unit = 5 cm; spacing between contour levels is 0.2)
   a. Use the contour plot to determine whether f_x and f_y are > 0, = 0, or < 0 at (1, 1.5) and (1.2, 0.6).
   b. The function plotted on the figure is $f(x,y)=x^3-xy^2+x^2y-4x^2+3x$. Calculate the actual values of the partial derivatives at (1, 1.5) and (1.2, 0.6).
   c. In the diagram, there are three places where the tangent plane is horizontal. Find the exact coordinates of each of these and characterize each as a max, a min or neither.



5. Consider the curve $\vec{r}(t)=\langle \cos 2t, \sin 2t, 2t^{3/2}\rangle$
   a. Find the length of the curve from $0\le t\le 1$
   b. Find the curvature as a function of t.
   c. Find the tangential and normal components of the acceleration.

6. Consider $g(x,y,z)=x^2+yz$
   a. Find a unit vector in the direction from (2, –1, 1) in which g decreases most rapidly.
   b. Find a parameterization of the line from that point in the direction of most rapid decrease.

7. Let $f(x,y)=xy^2+x+2y$ and $g(x,y)=xy-1$. Use the method of Lagrange multipliers to find the minimum and maximum values, if they exist, of $f(x,y)$ subject to the constraint $g(x,y)=0$ with $x > 0$. In the case that they do exist, identify all of the points (x, y) at which these values are attained.

8. Find the area of the part of the saddle $z=x^2-y^2$ that lies inside the cylinder $x^2+y^2=4$.

9. Let $f(x,y)=x^3y+y^3$, and C be $y^2 = x$, between (1, –1) and (1,1), directed upwards.

   a. Calculate $\vec{F}=\overrightarrow{\nabla f}$.

   b. Calculate the integral $\int_C \vec{F}\cdot\overrightarrow{dr}$ three different ways:

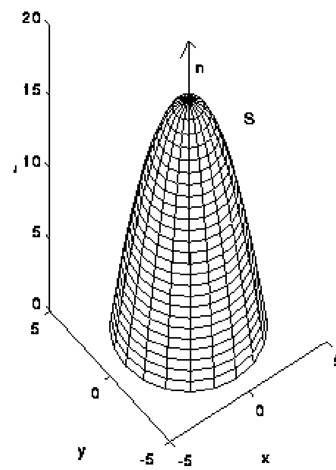
      i. directly;

      ii. by using path-independence to replace C by a simpler path.

      iii. by using the Fundamental Theorem for line integrals.

10. Verify Green's theorem in the normal form, i.e. $\oint_C \vec{F}\cdot\hat{n}\,ds=\iint_D \operatorname{div}\vec{F}(x,y)\,dA$, by calculating both sides and showing they are equal if $\vec{F}=\langle x^2,xy\rangle$ and C is the square with opposite vertices at (0,0) and (1,1).

11. Verify Stokes' theorem for the paraboloid $z=16-x^2-y^2$ for $z\ge 0$ and the vector field $F=\langle 3y,4z,-6x\rangle$. You may find it convenient to use polar coordinates to evaluate the surface integral of the curl.



12. Use the divergence theorem to calculate $\iiint_D 1dV$ where V is the region bounded by the cone $z=\sqrt{x^2+y^2}$ and the plane z = 1. To do this, you will need a simple field whose divergence is `1. How about $\vec{F}=\langle x,0,0\rangle$? Hint: You can parameterize the cone by $\vec{r}(t,\theta)=\langle t\cos\theta,t\sin\theta,t\rangle$.

13. The position of a particle moving in the plane at t is given by $\vec{r}(t)=\left\langle \frac{1}{\sqrt{1+t^2}},\frac{t}{\sqrt{1+t^2}}\right\rangle$.

    a. Find the velocity of the particle at time t.

    b. Find the speed of the particle at time t.

    c. What is the particle's highest speed and when does that occur?

14. Find equations of the normal plane and the osculating plane for the curve $\vec{r}(t)=\langle t,t^2,t^3\rangle$ at (1,1,1)

15. Consider the curve $\vec{R}(\theta)=\left\langle \int_0^\theta \cos\left(\frac{\pi t^2}{2}\right)dt,\ \int_0^\theta \sin\left(\frac{\pi t^2}{2}\right)dt\right\rangle$

    a. Show that the arc length along the curve from $\vec{R}(0)$ to $\vec{R}(\theta)$ is θ.

    b. Find the curvature at θ.

16. Find the direction in which $f(x, y, z) = xe^{yz}$ increases most rapidly at the point $(2,1,0)$. What is the maximum rate of increase?

17. Find the minimum value of $f(x, y) = e^{x/2}(x + y^2)$ and use the second derivative test: $f_{xx}f_{yy} - (f_{xy})^2 > 0$ to verify this.

18. Describe and sketch a graph for the solid whose volume is given by the following integral: $\int_0^{2\pi}\int_{5\pi/6}^{\pi}\int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ and evaluate the integral.

19. Using cylindrical coordinates, set up an integral for the volume under the paraboloid $z = r^2$ and above the disc $0 \le r \le \cos\theta$.

20. Find the work done by the force field $\vec{F}(x, y, z) = \langle y, z, x \rangle$ in moving a particle along the helix $\vec{r}(t) = \langle \sin t, \cos t, t \rangle$ for $0 \le t \le \frac{\pi}{2}$.

21. Verify Stoke's theorem for the vector field $\vec{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle$ where S is the part of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane.

22. Let $\vec{F}(x, y, z) = \langle z \tan^{-1}(y^2), z^3 \ln(x^2 + 1), z \rangle$. Find the flux of $\vec{F}$ across the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$ and is oriented upwards.

1. Given points $A(1,0,1)$, $B(2,3,0)$, $C(-1, 1, 4)$ and $D(0,3,2)$, find the volume of the parallelepiped with adjacent edges AB, AC, and AD. Hint: The volume of the parallelepiped determined by vectors $\vec{a},\vec{b},\vec{c}$ is the triple product, $\left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right|$.

   SOLN: Take $\vec{a}=\overrightarrow{AB}=\langle 1,3,-1\rangle$, $\vec{b}=\overrightarrow{AC}=\langle -2,1,3\rangle$ and $\vec{c}=\overrightarrow{AD}=\langle -1,3,1\rangle$ so that

   $\left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right|=\left|\langle 1,3,-1\rangle\cdot\langle -8,-1,-5\rangle\right|=\left|-8-3+5\right|=6$

2. Consider the lines $\vec{r}_1(t)=\langle 1+2t,3+4t,5+6t\rangle$ and $\vec{r}_2(t)=\langle 4t-5,3t+1,2t+7\rangle$.
   a. Show that these lines intersect by finding the point of intersection.
      SOLN: $\vec{r}_1(1)=\vec{r}_2(2)=\langle 3,7,11\rangle$
   b. Find an equation for the plane containing these lines.
      SOLN: Take one vector along r_1; $\vec{r}_1(2)-\vec{r}_1(1)=\langle 5,11,17\rangle-\langle 3,7,11\rangle=\langle 2,4,6\rangle=2\langle 1,2,3\rangle$ and another along r_2: $\vec{r}_2(3)-\vec{r}_2(2)=\langle 7,10,13\rangle-\langle 3,7,11\rangle=\langle 4,3,2\rangle$, whence a normal to the plane is given by $\langle 4,3,2\rangle\times\langle 1,2,3\rangle=\langle 5,-10,5\rangle=5\langle 1,-2,1\rangle$, so that the equation of the plane is $x-2y+z=0$

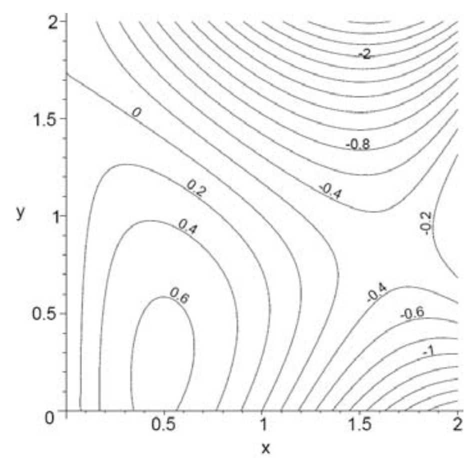
3. Let $z=x^y\left(=e^{y\ln x}\right)$. Give an approximate formula for the small change Δz that results from small changes Δx and Δy in the values of x and y. Use this to approximate the value of $1.99^{3.02}$.

   SOLN: $z=e^{y\ln x}$, so $\dfrac{\partial z}{\partial x}=\dfrac{y}{x}e^{y\ln x}=yx^{y-1}$ and $\dfrac{\partial z}{\partial y}=(\ln x)e^{y\ln x}=x^y\ln x$. Thus, near a point (x_0, y_0), with $\Delta x=x-x_0$ and $\Delta y=y-y_0$ we have $\Delta z\approx y_0x_0^{y_0-1}\Delta x+x_0^{y_0}\ln x_0\Delta y$. Thus

   $1.99^{3.02}=2^3+\Delta z\approx 8+(3)2^2(-0.01)+8\ln 2(0.02)=8-0.12+0.16\ln 2\approx 7.9909$

   To be sure, this estimate is slightly above the TI86 approximation $1.99^{3.02}\approx 7.9898$

4. The figure at right is the contour plot of a function of two variables, $f(x,y)$, for x and y ranging from 0 to 2. The scale is 1 unit = 5 cm; spacing between contour levels is 0.2)

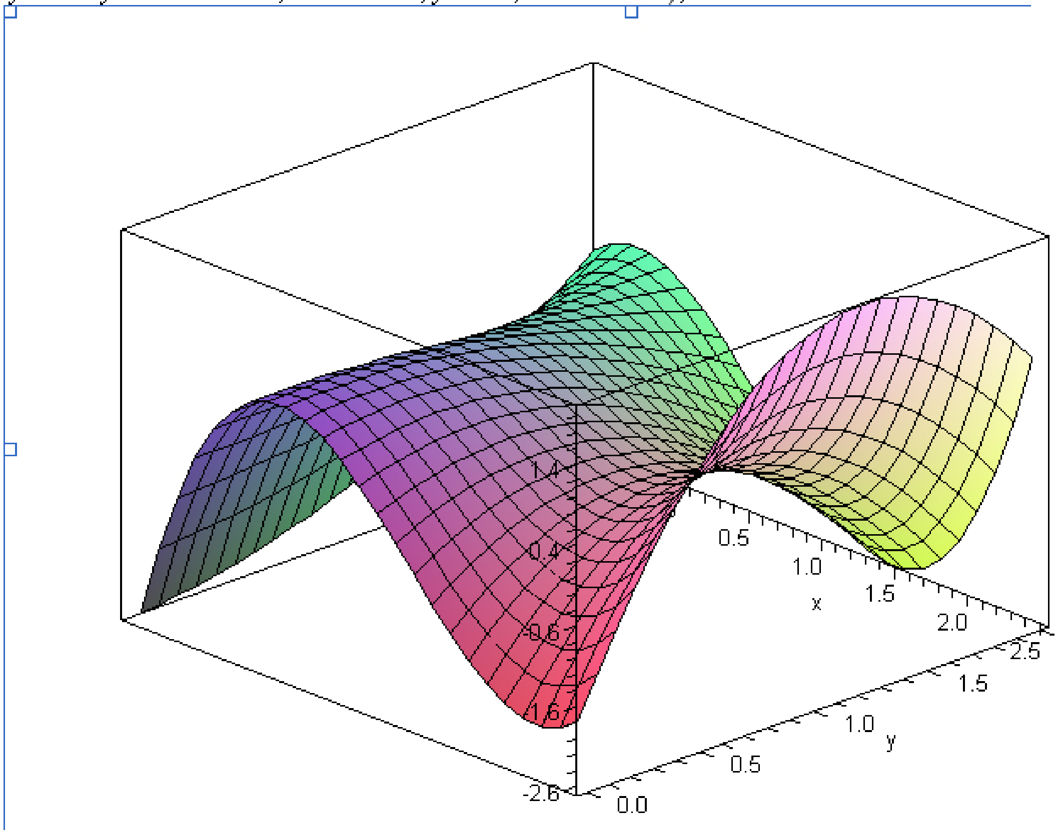


   a. Use the contour plot to determine whether f_x and f_y are > 0, = 0, or < 0 at (1, 1.5) and (1.2, 0.6).
      SOLN: $f_x(1,1.5)<0$ and $f_y(1,1.5)<0$ whereas
      $f_x(1.2,0.6)<0$ and $f_y(1.2,0.6)=0$
   b. The function is $f(x,y)=x^3-xy^2+x^2y-4x^2+3x$.
      Calculate the actual values of the partial derivatives at (1, 1.5) and (1.2, 0.6).
      SOLN: $f_x(x,y)=3x^2-y^2+2xy-8x+3$ and $f_y(x,y)=-2xy+x^2$ whence
      $f_x(1,1.5)=3-2.25+3-8+3=-1.25$, $f_y(1,1.5)=-3+1=-2$
      $f_x(1.2,0.6)=4.32-0.36+2.16-9.6+3=-1.2$, $f_y(1.2,0.6)=-1.44+1.44=0$

c. In the diagram, there are three places where the tangent plane is horizontal. Find the exact coordinates of each of these and characterize each as a max, a min or neither.

SOLN: First find critical points where $f_x(x,y)=0$ and $f_y(x,y)=0$. For the latter, $-2xy+x^2=0 \Leftrightarrow x=0$ or $x=2y$. Now $x = 0$ means that $f_x(x,y)=0 \Rightarrow y=\sqrt{3}$ and $x = 2y$ also means that $f_x(2y,y)=15y^2-16y+3=0 \Leftrightarrow y=\dfrac{16\pm\sqrt{256-120}}{30}=\dfrac{8\pm\sqrt{34}}{15}$. Thus there are three critical points: $\left(0,\sqrt{3}\right)$ and $\left(\dfrac{8\pm\sqrt{34}}{30},\dfrac{8\pm\sqrt{34}}{15}\right)$. It is evident from the level curves plot that $\left(\dfrac{8-\sqrt{34}}{30},\dfrac{8-\sqrt{34}}{15}\right)$ is a local max and $\left(\dfrac{8+\sqrt{34}}{30},\dfrac{8+\sqrt{34}}{15}\right)$ is a saddle. The point $\left(0,\sqrt{3}\right)$ on the y-axis is not so obvious so we look at the discriminant: $f_{xx}(x,y)=6x+2y-8$, $f_{yy}(x,y)=-2x$ and $f_{xy}(x,y)=-2y+2x$ so that at $\left(0,\sqrt{3}\right)$, $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2\sqrt{3} & -2\sqrt{3} \\ -2\sqrt{3} & 0 \end{vmatrix} = -12<0 \Rightarrow$ this is also a saddle. I thought it'd be.

$plot3d\left(x^3 - x\cdot y^2 + x^2\cdot y - 4\cdot x^2 + 3\cdot x, x=-0.5..2.5, y=0..2, axes=boxed\right);$



5. Consider the curve $\vec{r}(t)=\left\langle \cos 2t, \sin 2t, 2t^{3/2}\right\rangle$

a. Find the length of the curve from $0\le t\le 1$

SOLN: $\int_0^1 \left|\vec{r}\,'(t)\right| dt = \int_0^1 \sqrt{4\sin^2 2t+4\cos^2 2t+9t}\,dt = \int_0^1 \sqrt{4+9t}\,dt = \dfrac{2}{27}(4+9t)^{3/2}\Big|_0^1 = \dfrac{26\sqrt{13}-16}{27}$

b. Find the curvature as a function of t.

$$\kappa(t)=\left|\frac{d\hat{T}}{ds}\right|=\left|\frac{d\hat{T}/dt}{ds/dt}\right|=\frac{\left|\vec{r}\,'(t)\times\vec{r}\,''(t)\right|}{\left|\vec{r}\,'(t)\right|^3}$$

SOLN:
$$=\frac{\left|\left\langle -2\sin(2t),2\cos(2t),3t^{1/2}\right\rangle\times\left\langle -4\cos(2t),-4\sin(2t),\frac{3}{2}t^{-1/2}\right\rangle\right|}{\left|\left\langle -2\sin(2t),2\cos(2t),3t^{1/2}\right\rangle\right|^3}$$
$$=\frac{\left|\left\langle 3t^{-1/2}\cos(2t)+12t^{1/2}\sin(2t),3t^{-1/2}\sin(2t)-12t^{1/2}\cos(2t),8\right\rangle\right|}{(4+9t)^{3/2}}$$
$$=\frac{\sqrt{9t^{-1}+64+144t}}{(4+9t)^{3/2}}=\frac{\sqrt{144t^2+64t+9}}{t^{1/2}(4+9t)^{3/2}}$$

c. Find the tangential and normal components of the acceleration.

SOLN: $\vec{a}_N=\dfrac{\left|\vec{r}\,'(t)\times\vec{r}\,''(t)\right|}{\left|\vec{r}\,'(t)\right|}=\dfrac{\sqrt{144t^2+64t+9}}{t^{1/2}(4+9t)^{1/2}}$,

$$\vec{a}_T=\frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{\left|\vec{r}\,'(t)\right|}=\frac{\left\langle -2\sin(2t),2\cos(2t),3t^{1/2}\right\rangle\times\left\langle -4\cos(2t),-4\sin(2t),\frac{3}{2}t^{-1/2}\right\rangle}{t^{1/2}(4+9t)^{1/2}}$$
$$=\frac{3}{\left(4t+9t^2\right)^{1/2}}$$

6. Consider $g(x,y,z)=x^2+yz$

a. Find a unit vector in the direction from (2, –1, 1) in which g decreases most rapidly.

SOLN: $-\nabla g(2,-1,1)=-\left\langle 2x,z,y\right\rangle\big|_{(2,-1,1)}=\left\langle -4,-1,1\right\rangle\Rightarrow\hat{u}=\dfrac{\left\langle -4,-1,1\right\rangle}{3\sqrt{2}}$.

b. Find a parameterization of the line from that point in the direction of most rapid decrease.

SOLN: $\vec{r}(t)=\left\langle 2,-1,1\right\rangle+t\left\langle -4,-1,1\right\rangle=\left\langle 2-4t,-1-t,1+t\right\rangle$

7. Let $f(x,y)=xy^2+x+2y$ and $g(x,y)=xy-1$. Use the method of Lagrange multipliers to find the minimum and maximum values, if they exist, of $f(x,y)$ subject to the constraint $g(x,y)=0$ with $x > 0$. In the case that they do exist, identify all of the points (x, y) at which these values are attained.

SOLN: At the optimal point we require that the normals are parallel:

$\nabla f(x,y)=\lambda\nabla g(x,y)\Leftrightarrow\left\langle y^2+1,2xy+2\right\rangle=\lambda\left\langle y,x\right\rangle$

Also, the constraint must be met. Thus we have three equations in three unknowns:

$$y^2+1=\lambda y$$
$$2xy+2=\lambda x$$
$$xy-1=0$$

From the last equation we substitute $xy = 1$ into the second equation and get $\lambda x = 4$ whence $y = \lambda/4$ and substituting into the first equation yields $\left(\dfrac{\lambda}{4}\right)^2+1=\dfrac{\lambda^2}{4}\Leftrightarrow\lambda=\pm\sqrt{\dfrac{16}{3}}=\pm\dfrac{4\sqrt{3}}{3}$ so that $y=\pm\dfrac{\sqrt{3}}{3}$

and, correspondingly, $x=\pm\sqrt{3}$. $f\left(\sqrt{3},\frac{\sqrt{3}}{3}\right)=2\sqrt{3}$ is a max and $f\left(-\sqrt{3},\frac{-\sqrt{3}}{3}\right)=-2\sqrt{3}$ is a min.

8. Find the area of the part of the saddle $z = x^2 - y^2$ that lies inside the cylinder $x^2 + y^2 = 4$.
SOLN: $dS=\sqrt{1+\left(z_x\right)^2+\left(z_y\right)^2}\,dA=\sqrt{1+4\left(x^2+y^2\right)}dA$ so that $A =$

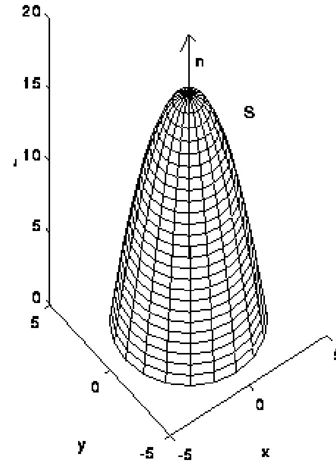
$$\iint_S 1\,dS=\int_0^{2\pi}\int_0^2 r\sqrt{1+4r^2}\,dr\,d\theta=2\pi\int_0^2 r\left(1+4r^2\right)^{1/2}dr=2\pi\left(\frac{2}{3}\left(\frac{1}{8}\right)\left(1+4r^2\right)^{3/2}\right)\Bigg|_0^2=\frac{17\pi}{6}\sqrt{17}-\frac{\pi}{6}$$

9. Let $f\left(x,y\right)=x^3y+y^3$, and C be $y^2 = x$, between (1, –1) and (1,1), directed upwards.
   a. Calculate $\vec{F}=\overrightarrow{\nabla f}$.
      SOLN: $\vec{F}=\overrightarrow{\nabla f}=\left\langle 3x^2y,x^3+3y^2\right\rangle$
   b. Calculate the integral $\int_C \vec{F}\cdot\overrightarrow{dr}$ three different ways:
      i. directly; $\int_C \vec{F}\cdot\overrightarrow{dr}=\int_{-1}^1\left\langle 3t^5,t^6+3t^2\right\rangle\left\langle 2t,1\right\rangle dt=\int_{-1}^1 7t^6+3t^2dt=t^7+t^3\Big|_{-1}^1=4$
      ii. by using path-independence to replace C by a simpler path.
      SOLN: The simpler path would be $\vec{r}\left(t\right)=\left\langle 1,t\right\rangle$ as t goes from –1 to 1.
      $\int_C \vec{F}\cdot\overrightarrow{dr}=\int_{-1}^1\left\langle t,1+3t^2\right\rangle\left\langle 0,1\right\rangle dt=\int_{-1}^1 1+3t^2dt=t+t^3\Big|_{-1}^1=4$
      iii. by using the Fundamental Theorem for line integrals.
      SOLN: $f\left(1,1\right)-f\left(1,-1\right)=2-\left(-2\right)=4$

10. Verify Green's theorem in the normal form, i.e. $\oint_C \vec{F}\cdot\hat{n}\,ds=\iint_D \operatorname{div}\vec{F}\left(x,y\right)dA$, by calculating both sides and showing they are equal if $\vec{F}=\left\langle x^2,xy\right\rangle$ and C is the square with opposite vertices at (0,0) and (1,1). Parametrize the four edges like so:.
$\vec{r}_1\left(t\right)=\left\langle t,0\right\rangle;\vec{r}_2\left(t\right)=\left\langle 1,t\right\rangle;\vec{r}_3\left(t\right)=\left\langle 1-t,1\right\rangle;\vec{r}_4\left(t\right)=\left\langle 0,1-t\right\rangle$ whence
$\vec{r}_1\,'\left(t\right)=\left\langle 1,0\right\rangle;\vec{r}_2\,'\left(t\right)=\left\langle 0,1\right\rangle;\vec{r}_3\,'\left(t\right)=\left\langle -1,0\right\rangle;\vec{r}_4\,'\left(t\right)=\left\langle 0,-1\right\rangle$ and the normal components for these are
$\vec{n}_1\left(t\right)=\left\langle 0,-1\right\rangle;\vec{n}_2\left(t\right)=\left\langle 1,0\right\rangle;\vec{n}_3\left(t\right)=\left\langle 0,1\right\rangle;\vec{n}_4\left(t\right)=\left\langle -1,0\right\rangle$
SOLN:

$$\oint_C \vec{F}\cdot\hat{n}\,ds=\int_0^1 0dt+\int_0^1 1dt+\int_0^1\left(1-t\right)dt+\int_0^1 0dt=1+1-\frac{1}{2}=\boxed{\frac{3}{2}}=\int_0^1\int_0^1 3x\,dx\,dy=\iint_D \operatorname{div}\vec{F}\left(x,y\right)dA$$

11. Verify Stokes' theorem for the paraboloid $z = 16 - x^2 - y^2$ for $z \ge 0$ and the vector field $F = \langle 3y, 4z, -6x \rangle$. You may find it convenient to use polar coordinates to evaluate the surface integral of the curl.



SOLN: If $z = 0$, then we have $r = 4$, which can be parameterized by $\vec{r}(t) = \langle 4\cos t, 4\sin t, 0 \rangle$ so that

$$\oint_C \vec{F} \cdot dr = \int_0^{2\pi} \langle 12\sin t, 0, -24\cos t \rangle \cdot \langle -4\sin t, 4\cos t, 0 \rangle dt$$

$$= -48\int_0^{2\pi} \sin^2 t dt = -48\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\Bigg|_0^{2\pi} = -48\pi$$

On the other side,

$$\iint_S \vec{\nabla} \times \vec{F} \cdot \overrightarrow{dS} = \int_0^{2\pi}\int_0^4 \langle -4, 6, -3 \rangle \cdot \langle 2r\cos\theta, 2r\sin\theta, 1 \rangle r dr d\theta = \int_0^{2\pi}\int_0^4 12r^2\sin\theta - 8r^2\cos\theta - 3r dr d\theta$$

$$= \int_0^{2\pi} 4r^3\sin\theta - \frac{8}{3}r^3\cos\theta - \frac{3}{2}r^2\Big|_0^4 d\theta = \int_0^{2\pi} 256\sin\theta - \frac{512}{3}\cos\theta - 24 d\theta$$

$$= -256\cos\theta - \frac{512}{3}\sin\theta - 24\theta\Big|_0^{2\pi} = -48\pi$$

12. Use the divergence theorem to calculate $\iiint_D 1 dV$ where V is the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1$. To do this, you will need a simple field whose divergence is `1. How about $\vec{F} = \langle x, 0, 0 \rangle$? Hint: You can parameterize the cone by $\vec{r}(t,\theta) = \langle t\cos\theta, t\sin\theta, t \rangle$.

SOLN: $\iiint_D 1 dV = \frac{\pi}{3}$ is the volume of the cone. To apply Guass' theorem, compute $\oiint_S \vec{F} \cdot \overrightarrow{dS} = \oiint_S \langle x, 0, 0 \rangle \cdot \overrightarrow{dS} = \iint_{S_1} \langle x, 0, 0 \rangle \cdot \overrightarrow{dS} + \iint_{S_2} \langle x, 0, 0 \rangle \cdot \overrightarrow{dS}$ where S_1 is the flat top of the cone and S_2 is the curved surface of the cone. Now since the normal to flat part of the surface is perpendicular to the field lines, the first integral is zero. For the second, we have the normal to the surface, $\vec{r}_t \times \vec{r}_\theta = \langle \cos\theta, \sin\theta, 1 \rangle \times \langle -t\sin\theta, t\cos\theta, 0 \rangle = \langle -t\cos\theta, -t\sin\theta, t \rangle$ which points upwards when we want something pointing outwards. So we negate it and integrate

$$\iint_{S_2} \langle x, 0, 0 \rangle \cdot \overrightarrow{dS} = \int_0^{2\pi}\int_0^1 \langle t\cos\theta, 0, 0 \rangle \cdot \langle t\cos\theta, t\sin\theta, -t \rangle dt d\theta = \int_0^{2\pi} \cos^2\theta d\theta \int_0^1 t^2 dt = \frac{\pi}{3}$$

13. The position of a particle moving in the plane at t is given by $\vec{r}(t) = \left\langle \frac{1}{\sqrt{1+t^2}}, \frac{t}{\sqrt{1+t^2}} \right\rangle$.

a. Find the velocity of the particle at time t.

SOLN: $\vec{r}\,'(t) = \left\langle \frac{-t}{(1+t^2)^{3/2}}, \frac{(1+t^2)^{1/2} - t^2(1+t^2)^{-1/2}}{(1+t^2)^{3/2}} \right\rangle = \left\langle \frac{-t}{(1+t^2)^{3/2}}, \frac{1}{(1+t^2)^{3/2}} \right\rangle$

b. Find the speed of the particle at time t.

SOLN: $|\vec{r}\,'(t)| = \sqrt{\frac{t^2+1}{(1+t^2)^3}} = \frac{1}{1+t^2}$

c. What is the particle's highest speed and when does that occur?
The speed is maximum of 1 when $t = 0$.

14. Find equations of the normal plane and the osculating plane for the curve $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at (1,1,1)

SOLN: $\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$ and $\vec{r}\,''(t) = \langle 0, 2, 6t \rangle$ so $\hat{T} = \langle 1, 2t, 3t^2 \rangle / \sqrt{1+4t^2+9t^4}$ and

$$\hat{T}\,' = \left\langle \frac{-2t(9t^2+2)}{(1+4t^2+9t^4)^{3/2}}, \frac{-2(9t^4-1)}{(1+4t^2+9t^4)^{3/2}}, \frac{6t(2t^2+1)}{(1+4t^2+9t^4)^{3/2}} \right\rangle$$ so at (1,1,1) the tangent vector is parallel to $\hat{T} = k_1 \langle 1,2,3 \rangle$ and the normal vector is parallel to $\hat{N} = k_2 \langle -11, 8, 9 \rangle$ so that the binormal is $\hat{B} = k_1 k_2 \langle 1,2,3 \rangle \times \langle -11,8,9 \rangle = -6k_1k_2 \langle 1,7,-5 \rangle$ whence the normal plane is $x + 2y + 3z = 6$ and the osculating plane is $11x - 8y - 9z = -6$.

15. Express the curvature of the path $\vec{R}(\theta) = \left\langle \int_0^\theta \cos\left(\frac{\pi t^2}{2}\right) dt,\ \int_0^\theta \sin\left(\frac{\pi t^2}{2}\right) dt \right\rangle$ as a function of the directed distance s measured from $(0,0)$.

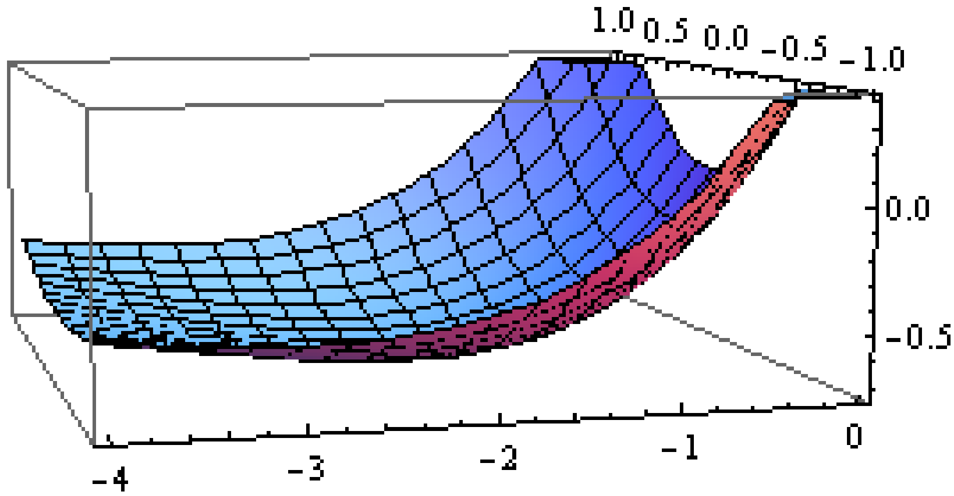
$$s(t) = \int_0^\theta \left|\vec{R}\,'(u)\right| du = \int_0^\theta \left| \cos^2\left(\frac{\pi u^2}{2}\right) + \sin^2\left(\frac{\pi u^2}{2}\right) \right| du = \int_0^\theta 1 du = \theta$$ so that the curvature is

$$\kappa = \left|\frac{d\hat{T}}{ds}\right| = \left|\frac{d\hat{T}}{d\theta}\right| = \left|\frac{d}{d\theta}\left\langle \int_0^\theta \cos\left(\frac{\pi t^2}{2}\right) dt,\ \int_0^\theta \sin\left(\frac{\pi t^2}{2}\right) dt \right\rangle\right| = \left|\left\langle \cos\left(\frac{\pi\theta^2}{2}\right),\ \sin\left(\frac{\pi\theta^2}{2}\right) \right\rangle\right| = 1$$

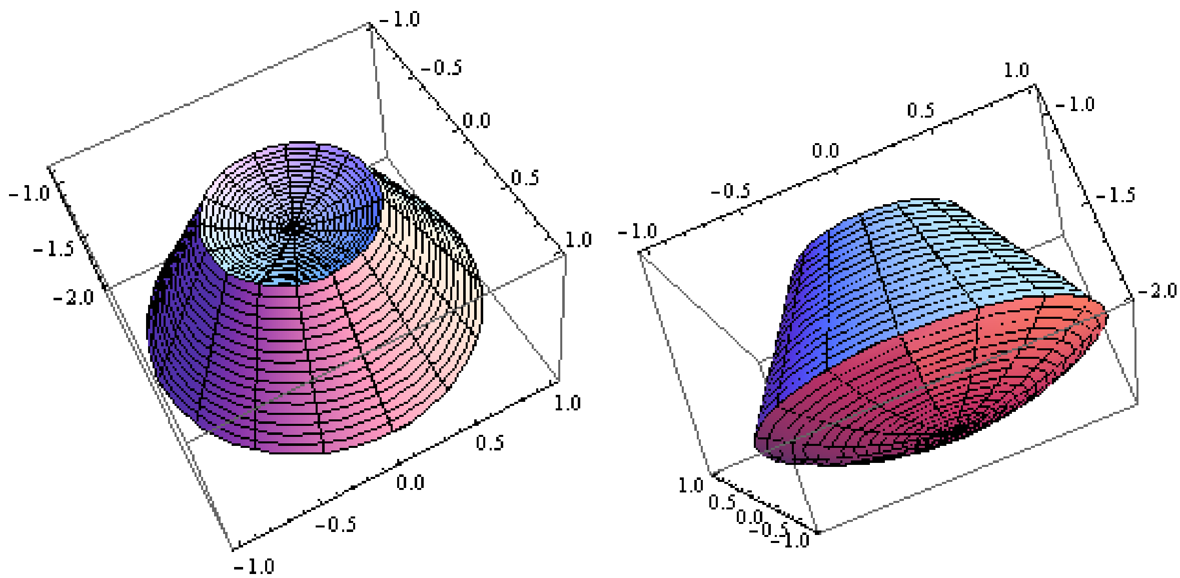
16. Find the direction in which $f(x,y,z) = xe^{yz}$ increases most rapidly at the point $(2,1,0)$.
What is the maximum rate of increase?
SOLN: The gradient vector is $\overline{\nabla f} = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$ which at (2,1,0) is $\langle 1,0,2 \rangle$.

17. Find the minimum value of $f(x,y) = e^{x/2}(x+y^2)$ and use the second derivative test: $f_{xx}f_{yy} - (f_{xy})^2 > 0$ to verify this.

SOLN: $f_x = \frac{e^{x/2}}{2}(x+y^2) + e^{x/2} = \frac{e^{x/2}}{2}(x+y^2+2) = 0 \Leftrightarrow x+y^2+2=0$ $f_y = 2ye^{x/2} = 0 \Leftrightarrow y = 0$ so the critical point is (-2,0), and since $f_{xx}f_{yy} - (f_{xy})^2 = \frac{e^x}{2}(x+y^2+4) - y^2e^x = \frac{e^x}{2}(x-y^2+4)$, which is $e^{-2} > 0$, we note that $f_{xx}(-2,0) = \frac{1}{2e} > 0$ which means $f(-2,0) = -\frac{2}{e}$ is a local minimum.



18. Describe and sketch a graph for the solid whose volume is given by the following integral:
$\int_0^{2\pi}\int_{-\pi}^{-5\pi/6}\int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ and evaluate the integral.
SOLN: This is the intersection of a cone whose top is at the origin and which opens downwards with an angle of $\pi/3$ with a spherical shell. Two views of this volume are shown below:



19. Using cylindrical coordinates, set up an integral for the volume under the paraboloid $z = r^2$ and above the disc $0 \le r \le \cos\theta$.
SOLN: $\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta} r^3 dr d\theta$

20. Find the work done by the force field $\overrightarrow{F}(x, y, z) = \langle y, z, x\rangle$ in moving a particle along the helix
$\vec{r}(t) = \langle \sin t, \cos t, t\rangle$ for $0 \le t \le \dfrac{\pi}{2}$.
SOLN: $\int_C \vec{F} \cdot \overrightarrow{dr} = \int_0^{\pi/2} \langle \cos t, t, \sin t\rangle \cdot \langle \cos t, -\sin t, 1\rangle dt = \int_0^{\pi/2} \cos^2 t + (1-t)\sin t \, dt = \dfrac{\pi}{4}$ …joules?

21. Verify Stoke's theorem for the vector field $\vec{F}(x,y,z)=\langle x^2,y^2,z^2\rangle$ where S is the part of the paraboloid $z=1-x^2-y^2$ above the xy-plane.

SOLN: Stokes' theorem states that

$$\oint_C \vec{F}\cdot\overrightarrow{dr}=\iint_{\partial C}\vec{\nabla}\times\vec{F}\cdot\overrightarrow{dS}$$

$$\oint_C \vec{F}\cdot\overrightarrow{dr}=\int_0^{2\pi}\langle\cos^2 t,\sin^2 t,0\rangle\cdot\langle-\sin t,\cos t,0\rangle dt=\int_0^{2\pi}-\sin t\cos^2 t+\sin^2 t\cos t\,dt=\left.\frac{\cos^3 t+\sin^3 t}{3}\right|_0^{2\pi}=0$$

This is consistent with $\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2 & y^2 & z^2\end{vmatrix}=\vec{0}$

22. Let $\vec{F}(x,y,z)=\left\langle z\tan^{-1}(y^2),z^3\ln(x^2+1),z\right\rangle$. Find the flux of $\vec{F}$ across the part of the paraboloid $z=x^2+y^2$ that lies below the plane z = 1 and is oriented upwards.

SOLN: The divergence theorem is helpful here. The outwards surface integral is

$$\oiint_S \vec{F}\cdot\overrightarrow{dS}=\iiint_E\vec{\nabla}\cdot\vec{F}dV=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{1-x^2-y^2}1dzdydx=\int_0^{2\pi}\int_0^1\int_0^{1-r^2}rdzdrd\theta=2\pi\int_0^1 r\left(1-r^2\right)dr$$

$$=2\pi\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{\pi}{2}$$