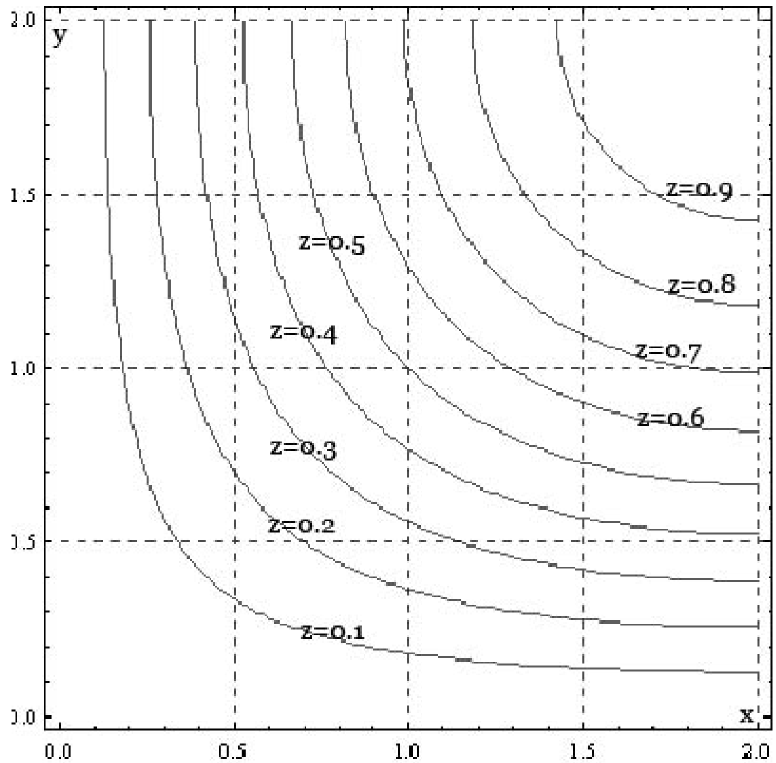
Math 2A – Vector Calculus – Chapter 14 Test – Fall '11   Name____________________________

Show your work. Don't use a calculator. Write responses on separate paper.

1. Consider the nice, smooth function $z = f(x, y)$ whose contour map is shown at right.



   a. Estimate function values to the nearest tenth to fill in the blank cells in the table below:

$x \backslash y$	0.4	1.0	1.5
0.2	0.05		0.14
0.4			
0.6	0.14		0.42

   b. Use the value in your table above to estimate $f_x(0.4,1.0)$ and $f_y(0.4,1.0)$ to the nearest tenth.

   c. Let $\vec{v}$ be the vector from $P(0.6,0.4)$ to $Q(0.2,1.5)$
   Compute $D_{\vec{v}} f(0.4,1.0)$ in two ways: as $\frac{\Delta z}{h}$ and as $\overrightarrow{\nabla z} \bullet \frac{\vec{v}}{|\vec{v}|}$

   d. Let $\vec{v}$ be the vector from $P(0.2,0.4)$ to $Q(0.6,1.5)$
   Compute $D_{\vec{v}} f(0.4,1.0)$ in two ways: as $\frac{\Delta z}{h}$ and as $\overrightarrow{\nabla z} \bullet \frac{\vec{v}}{|\vec{v}|}$.

2. Show that $\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$ does not exist.

3. Sketch level curves $f(x,y) = 0$, $f(x,y) = ½$ and $f(x,y) = 1/5$ for the function

$$f(x,y) = \begin{cases} \frac{x^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

4. Find points on the surface $xy + yz + zx - x - z^2 = 0$ where the tangent plane is parallel to the xy-plane.

5. Show that $f(x, y) = \arctan(y/x)$ satisfies the two dimensional Laplace equation, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

6. Find the direction in which $f(x,y) = x^2 + \cos(xy)$ increases most rapidly at the point $P_0(1,0)$.

7. Consider $f(x, y) = x^3 + 3xy + y^3$
   a. Find the critical points.

   b. Find all maxima, minima and saddle points and evaluate the function at those points.

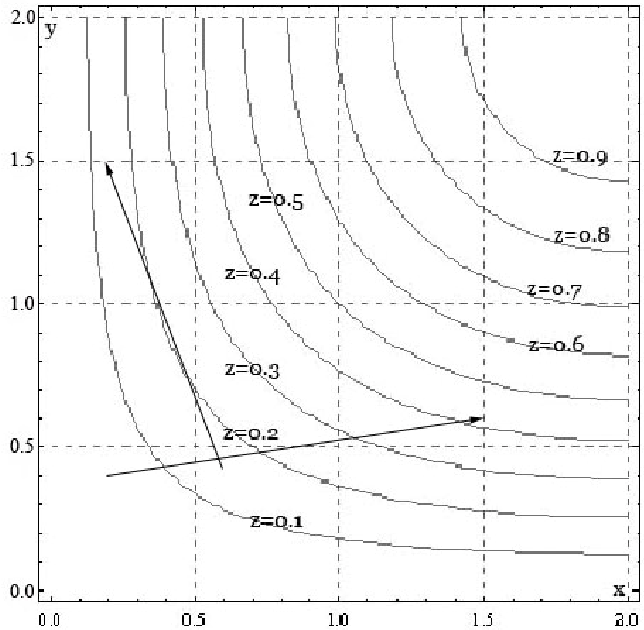
8. Find the absolute max. and min. values of $f(x, y) = xy$ on the ellipse $x^2 + 4y^2 = 8$ in two ways:
   a. By using the parameterization $\langle x, y\rangle = \langle 2\sqrt{2}\cos t, \sqrt{2}\sin t\rangle$

   b. By using Lagrange multipliers.

9. Find a level surface for the density function $f(x, y, z) = x^2 + y^2 - z^2$ that has the tangent plane $2x + 3y - z = 3$.

Math 2A – Vector Calculus – Chapter 11 Test Solutions – Fall '09

1. Consider the nice, smooth function $z = f(x,y)$ whose contour map is shown at right.



a) Estimate function values to the nearest tenth to fill in the blank cells in the table below:

$x \setminus y$	0.4	1.0	1.5
0.2	0.05	0.1	0.14
0.4	0.1	0.2	0.3
0.6	0.14	0.3	0.42

b) Use the value in your table above to estimate $f_x(0.4,1.0)$ and $f_y(0.4,1.0)$ to the nearest tenth.

$$f_x(0.4,1.0) \approx \frac{\Delta z}{\Delta x} = \frac{0.3-0.1}{0.6-0.2} = \frac{1}{2}$$

$$f_y(0.4,1.0) \approx \frac{\Delta z}{\Delta y} = \frac{0.3-0.1}{1.5-0.4} = \frac{2}{11}$$

c) Let $\vec{v}$ be the vector from $P(0.6,0.4)$ to $Q(0.2,1.5)$

Compute $D_{\vec{v}} f(0.4,1.0)$ in two ways: as $\frac{\Delta z}{h}$ and as $\overrightarrow{\nabla z} \bullet \frac{\vec{v}}{|\vec{v}|}$

SOLN: $|\vec{v}| = |\overrightarrow{PQ}| = |\langle 0.2-0.6, 1.5-0.4\rangle| = \sqrt{0.16+1.21} = \sqrt{1.37} \approx 1.17$ or 1.2 will do.

$$\boxed{D_{\vec{v}} f(0.4,1.0) \approx \frac{\Delta z}{h} = \frac{0}{h} = 0} \text{ or}$$

$$\overrightarrow{\nabla z} \bullet \frac{\vec{v}}{|\vec{v}|} = \langle f_x, f_y \rangle \bullet \frac{\langle -0.4, 1.1\rangle}{1.2} = \langle 0.5, 0.18\rangle \bullet \frac{\langle -0.4, 1.1\rangle}{1.2} \approx \boxed{\frac{-0.20+0.20}{1.2} = 0}$$

d) Let $\vec{v}$ be the vector from $P(0.2,0.4)$ to $Q(0.6,1.5)$

Compute $D_{\vec{v}} f(0.4,1.0)$ in two ways: as $\frac{\Delta z}{h}$ and as $\overrightarrow{\nabla z} \bullet \frac{\vec{v}}{|\vec{v}|}$.

SOLN: $|\vec{v}| = |\overrightarrow{PQ}| = |\langle 0.6-0.2, 1.5-0.4\rangle| = \sqrt{0.16+1.21} = \sqrt{1.37} \approx 1.2$

$$\text{So, } \boxed{D_{\vec{v}} f(0.4,1.0) \approx \frac{\Delta z}{h} = \frac{0.37}{1.2} \approx 0.31} \text{ or}$$

$$\overrightarrow{\nabla z} \bullet \frac{\vec{v}}{|\vec{v}|} = \langle f_x, f_y \rangle \bullet \frac{\langle 0.4, 1.1\rangle}{1.2} = \langle 0.5, 0.18\rangle \bullet \frac{\langle 0.4, 1.1\rangle}{1.2} \approx \boxed{\frac{0.40}{1.2} \approx 0.33}$$

2. Show that $\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$ does not exist.

SOLN: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{\sqrt{x^2+y^2}} = \lim\limits_{r\to 0} \dfrac{r\cos\theta}{r} = \cos\theta$ is path dependent.

Also, along the line $y = x$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{\sqrt{x^2+y^2}} = \lim\limits_{x\to 0} \dfrac{x}{\sqrt{x^2+x^2}} = \lim\limits_{x\to 0} \dfrac{x}{\sqrt{2}\,|x|}$ does not exist because

$$\lim_{x\to 0^+} \frac{x}{\sqrt{2}\,|x|} = \lim_{x\to 0^+} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \neq \lim_{x\to 0^-} \frac{x}{\sqrt{2}\,|x|} = \lim_{x\to 0^+} \frac{-1}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

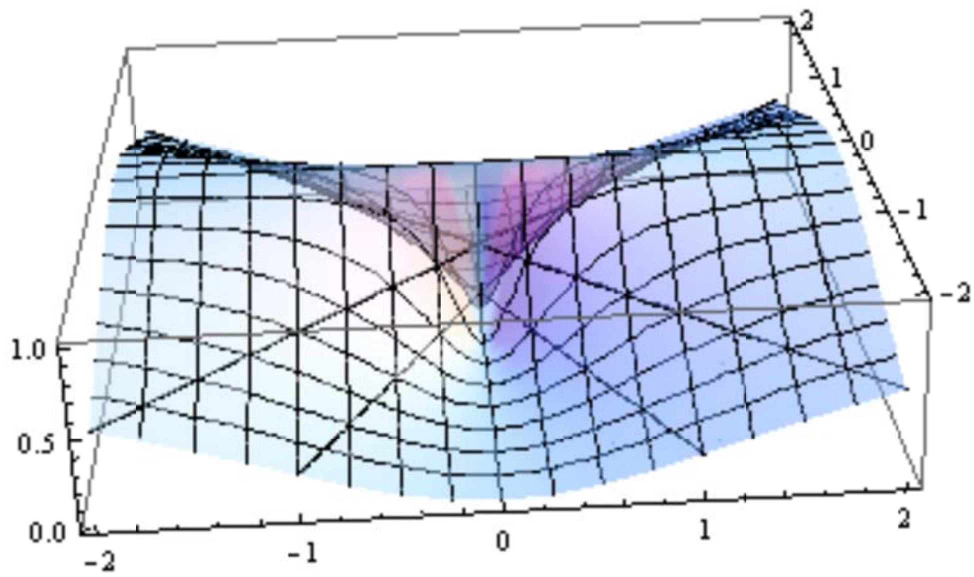
3. Sketch level curves $f(x,y) = 0$, $f(x,y) = ½$ and $f(x,y) = 1/5$ for the function

$$f(x,y) = \begin{cases} \dfrac{x^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

SOLN: $\dfrac{x^2}{x^2+y^2} = \dfrac{1}{2} \Leftrightarrow x^2 = y^2 \Leftrightarrow y = \pm x$ $\dfrac{x^2}{x^2+y^2} = \dfrac{1}{5} \Leftrightarrow 4x^2 = y^2 \Leftrightarrow y = \pm 2x$

The graphic below is constructed in Mathematica with
ParametricPlot3D[{{ , , ^2(^2+ ^2)},{ , ,1/2},{ ,− ,1/2},{− 2, ,1/5},{ 2, ,1/5}},
{ ,−2,2},{ ,−2,2},PlotStyle→Directive[Opacity[0.5],Thick]].

You see the two lines which form the intersection of the plane $z = ½$ and the lines where the plane $z = 1/5$ intersects the surface.



This shows, by the way, that there is no limiting value of z as (x,y) approaches $(0,0)$.

4. Find points on the surface $xy + yz + zx - x - z^2 = 0$ where the tangent plane is parallel to the xy-plane.
SOLN: This is a level surface for the function $f(x,y,z) = xy + yz + zx - x - z^2$, so the gradient vector $\overrightarrow{\nabla f} = \langle f_x, f_y, f_z \rangle = \langle y+z-1, x+z, y+x-2z \rangle$ is normal to the tangent plane. This means the tangent plane is horizontal only if $f_x = y + z - 1 = 0$ and

$f_y = x + z = 0$. Each of these equations describes a plane, and the intersection of these planes is the line $\vec{r}(t) = \langle -t, 1-t, t \rangle = \langle 0,1,0 \rangle - t\langle 1,1,-1 \rangle$.

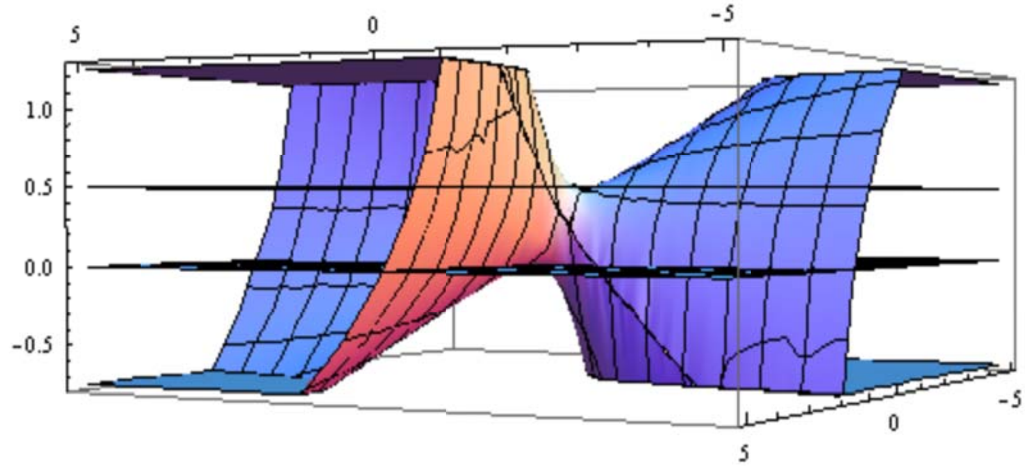
Plug $\vec{r}(t) = \langle -t, 1-t, t \rangle$ into $f(x,y,z) = f(-t,1-t,t) = -t(1-t) + (1-t)t - t^2 + t - t^2 = -t(2t-1) = 0$

so either $t = 0$ or $t = ½$ whence the points where tangent plane is horizontal are (0,1,0) and (–½, ½, ½). To help visualize what's going on here, you might solve the equation for the surface for z:

$$z^2 - (x+y)z + x(1-y) = 0 \Leftrightarrow z = \frac{x+y \pm \sqrt{(x+y)^2 - 4x(1-y)}}{2}.$$

We can then visualize this in Mathematica with the following command:

```
Plot3D[{{(x+y+Sqrt[(x+y)^2-4x(1-y)])/2},
        {(x+y-Sqrt[(x+y)^2-4x(1-y)])/2},
        {0},{1/2}},{x,-5,5},{y,-5,5}]
```



The graph shows a tilted hyperboloid of one sheet with the two tangent planes almost dead edge-on.

5. Show that $f(x,y) = \arctan(y/x)$ satisfies the two dimensional Laplace equation, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

$$\nabla f(x,y) = \left\langle \frac{d}{dx}\arctan(y/x), \frac{d}{dy}\arctan(y/x) \right\rangle$$

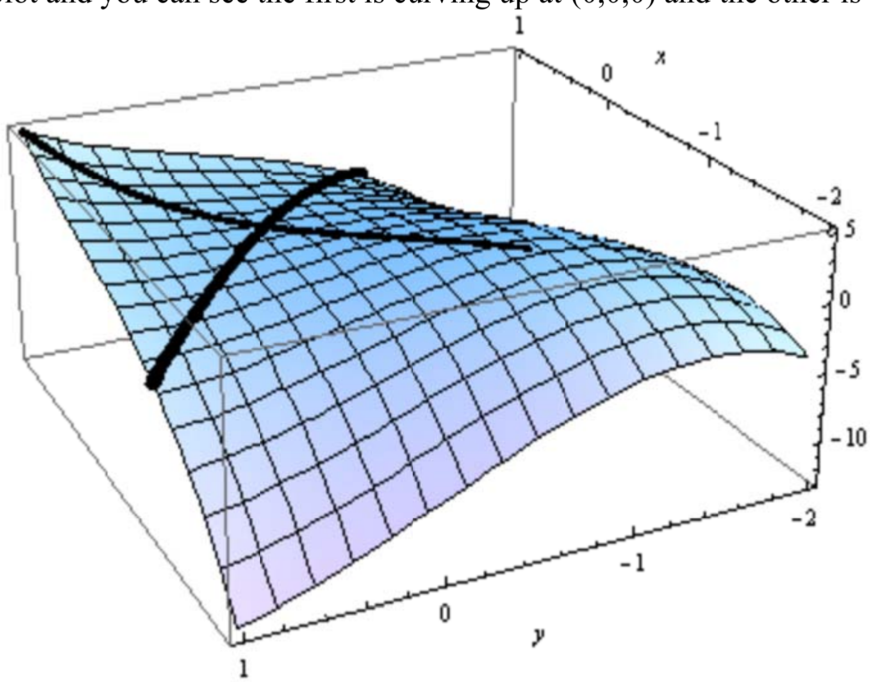
$$= \left\langle \frac{-y}{x^2\left(1+(y/x)^2\right)}, \frac{1}{x\left(1+(y/x)^2\right)} \right\rangle = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

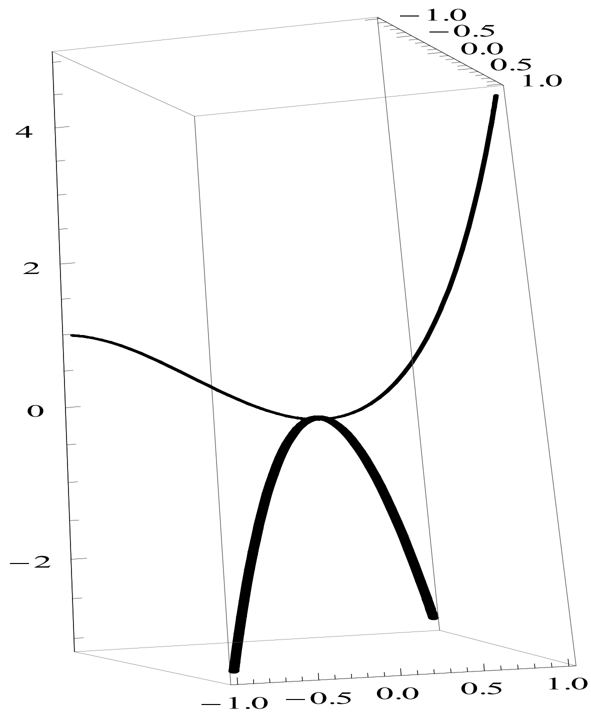
$$\nabla^2 f(x,y) = f_{xx} + f_{yy} = \frac{-2xy}{\left(x^2+y^2\right)^2} + \frac{2yx}{\left(x^2+y^2\right)^2} = 0$$

This is kind of interesting in polar form: $f\left(r\cos\theta, r\sin\theta\right) = \arctan\left(\tan\theta\right) = \begin{cases} \theta \\ \theta+\pi \end{cases}$

and $\nabla f\left(r,\theta\right) = \left\langle \frac{d}{dr}\arctan\left(\theta\right), \frac{d}{d\theta}\theta \right\rangle = \left\langle 0,1 \right\rangle$

6. Find the direction in which $f(x,y) = x^2 + \cos(xy)$ increases most rapidly at the point $P_0(1,0)$.
   SOLN: $\vec{\nabla} f\left(1,0\right) = \left\langle 2x - y\sin xy, -x\sin xy \right\rangle\Big|_{(1,0)} = \left\langle 2,0 \right\rangle$ so f grows at a rate of 2/1 in that direction.
7. Consider $f\left(x,y\right) = x^3 + 3xy + y^3$
   a) Find the critical points.
   SOLN: $f_x = 3x^2 + 3y = 0 \Leftrightarrow y = -x^2$ and $f_y = 3x + 3y^2 = 0 \Leftrightarrow x = -y^2$, substituting, we get $y = -\left(-y^2\right)^2 = -y^4 \Leftrightarrow y = 0$ or $y = -1$, leading to critical points (0,0) and (–1, –1).
   b) Find all maxima, minima and saddle points and evaluate the function at those points.
   SOLN: $D = f_{xx} f_{yy} - \left(f_{xy}\right)^2 = \left(6x\right)\left(6y\right) - 3^2 = 36xy - 9$ is positive at (–1, –1) where $f_{xx} = -6 < 0$, so this is a local maximum. The point (0,0) has $D < 0$ so the point is a saddle. In the diagram below, the local max at (-1,-1,1) seems evident. The saddle is a little more subtle. Look at the curves $\vec{r}_1 = \left\langle t, t, 2t^3 + 3t^2 \right\rangle$ and $\vec{r}_2 = \left\langle t, -t, -3t^2 \right\rangle$ are shown on the plot and you can see the first is curving up at (0,0,0) and the other is curving down.





8. Find the absolute max. and min. values of $f(x, y) = xy$ on the ellipse $x^2 + 4y^2 = 8$ in two ways
   a) By using the parameterization $\langle x, y\rangle = \langle 2\sqrt{2}\cos t, \sqrt{2}\sin t\rangle$

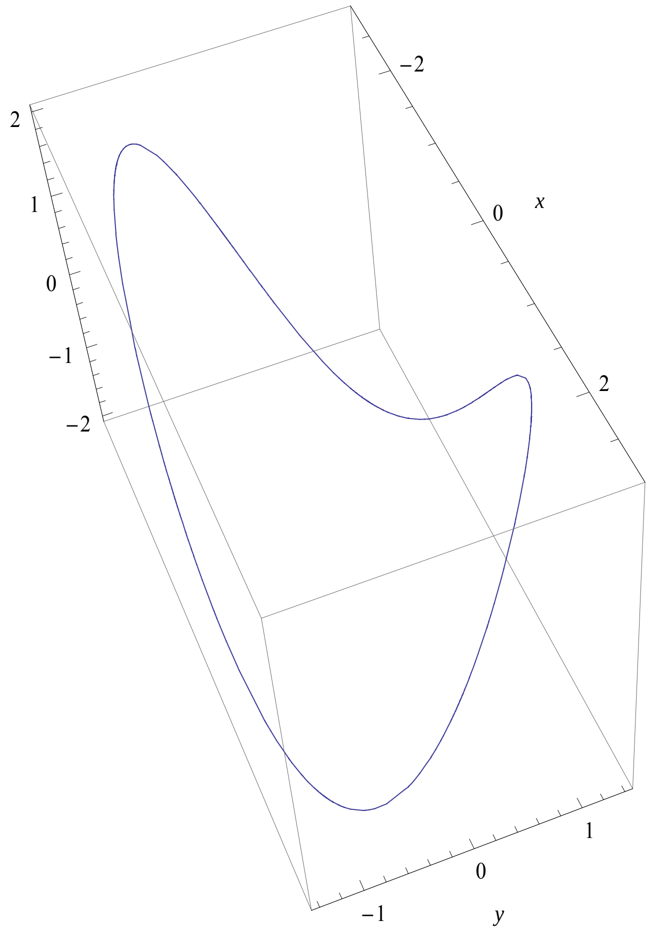
      SOLN: Along the path $\langle x, y\rangle = \langle 2\sqrt{2}\cos t, \sqrt{2}\sin t\rangle$, $z = 4\cos t \sin t = 2\sin 2t$ so $z' = 4\cos 2t = 0$
      if t = an odd multiple of $\pi/4$. At $\pi/4$ and $5\pi/4$ $z'' = -4$ so (2, 1, 2) and (–2,–1,2) are global maxima and at $3\pi/4$ and $7\pi/4$ $z'' = 4$ so (–2, 1, –2) and (2,–1, –2) are global minima.

   b) By using Lagrange multipliers.

      SOLN: $\nabla f = \lambda \nabla g \Leftrightarrow \langle y, x\rangle = \lambda\langle 2x, 8y\rangle$ leads to the system

$y = 2\lambda x$	Substituting from the second to the first, $y = 16\lambda^2 y$, we know that either
$x = 8\lambda y$	$y = 0$ or $\lambda = \pm 1/4$. If $y = 0$ then $x = 0$ and then constraint $x^2 + 4y^2 = 8$ can't
$x^2 + 4y^2 = 8$	be met so $\lambda = \pm 1/4$ which means $y = \pm x/2$ and substituting into the the
	ellipse equation, $x^2 + x^2 = 8 \Leftrightarrow x = \pm 2$ meaning that $y = \mp 1$

After investigating we determine that the global max is 2 occurring at (2,1) and (–2,–1) and the global min is –2 occurring at (–2,1) and (2,–1).



9. Find a level surface for the density function $f(x, y, z) = x^2 + y^2 - z^2$ that has the tangent plane $2x + 3y - z = 3$.
SOLN: The normal to the level surface will be parallel to the normal to the plane if $\overline{\nabla f} = \langle 2x, 2y, -2z\rangle = \lambda\langle 2,3,-1\rangle$ so that $\lambda = x = 2y/3 = 2z$ and substituting into the equation of the plane, $2\lambda + 9\lambda/2 - \lambda/2 = 6\lambda = 3$ or $\lambda = 1/2$ and thus $f(1/2, 3/4, 1/4) = (1/2)^2 + (3/4)^2 - (1/4)^2 = 3/4$. So the level surface is $x^2 + y^2 - z^2 = 3/4$