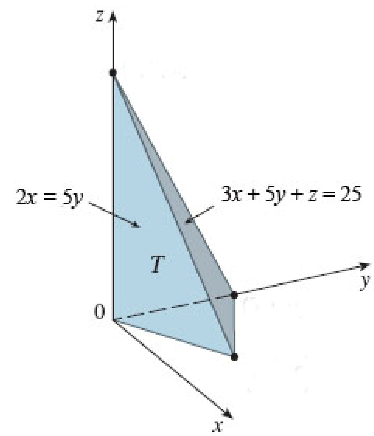
Math 2A – Multivariate Calculus – Chapter 15 Test – Fall ’11 Name________________________________

Show work for credit. Write all responses on separate paper. No calculators.

1. Find the volume the tetrahedron bounded by $2x = 5y$, $3x + 5y + z = 25$, $z = 0$ and $x = 0$ by setting up and evaluating an iterated integral. See the diagram at right.

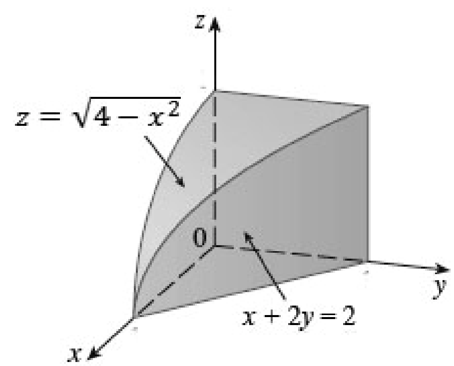



2. Sketch the region of integration for $\int_0^{\sqrt{3}} \int_{\arctan x}^{\pi/3} f(x,y)\,dydx$ and change the order of integration.

3. Define the improper integral
$$I = \iint\limits_{\mathbb{R}^2} e^{-(x^2+y^2)} dA = \lim_{a\to\infty} \iint\limits_{S_a} e^{-(x^2+y^2)} dA$$
where $S_a$ is the square with vertices $(-a,-a), (a,-a), (a,a)$ and $(-a,a)$.
Use this to show that $\int_{\infty}^{\infty} e^{-x^2} dx \int_{\infty}^{\infty} e^{-y^2} dy = \pi$

4. Evaluate $\iiint_E x dV$ is bounded by the cylinder $x + z^2 = 4$, and the planes $x + y = 4$, $x = 0$, $y = 0$, and $z = 0$.

5. Consider the density $\rho(x,y,z) = 1 + x^2 + y^2 + z^2$ over the volume bounded by $x^2 + z^2 = 4$, $x + 2y = 2$, $z = 0$, $y = 0$ and $x = 0$, as shown at right.
Set up but ***do not evaluate*** three iterated integrals to compute moments about the
a. *xy*-plane.
b. *xz*-plane.
c. *yz*-plane.

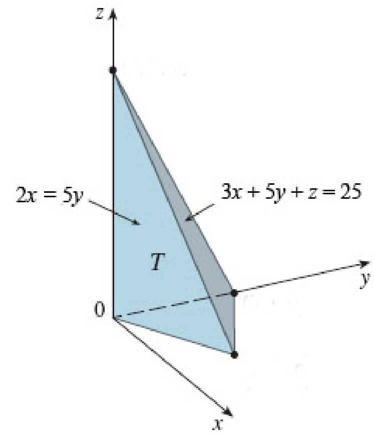



6. Use spherical coordinates to find the volume of the part of a sphere of $\rho = cos\phi$ that lies between the cones $\phi = \pi/4$ and $\phi = \pi/3$.

7. Determine the value of $\iint_D \sqrt{\frac{x+y}{x-2y}} dA$ where $D$ is the region in $\mathbb{R}^2$ enclosed by the lines $y = \frac{x}{2}$, $y = 0$, and $x + y = 1$. Hint: make a convenient change of variables.

# Math 2A – Multivariate Calculus – Chapter 15 Test Solutions.

1. Find the volume the tetrahedron bounded by $2x = 5y$, $3x + 5y + z = 25$, $z = 0$ and $x = 0$ by setting up and evaluating an iterated integral. See the diagram at right.




SOLN: The point of intersection of the planes with the *xy*-plane is the point satisfying the system $\begin{matrix} 3x + 5y = 25 \\ 2x - 5y = 0 \end{matrix}$ that is, $(x, y) = (5,2)$. To evaluate this as a single integral, do the bound for *y* last: $\frac{2x}{5} \le y \le \frac{25-3x-z}{5}$. Then you can evaluate either $\int_0^5 \int_0^{25-3x} \int_{\frac{2x}{5}}^{\frac{25-3x-z}{5}} dy\,dz\,dx$ or $\int_0^5 \int_0^{\frac{25-z}{3}} \int_{\frac{2x}{5}}^{\frac{25-3x-z}{5}} dy\,dx\,dz$. Other iterations mean splitting the integral into two parts. Here are the computations:

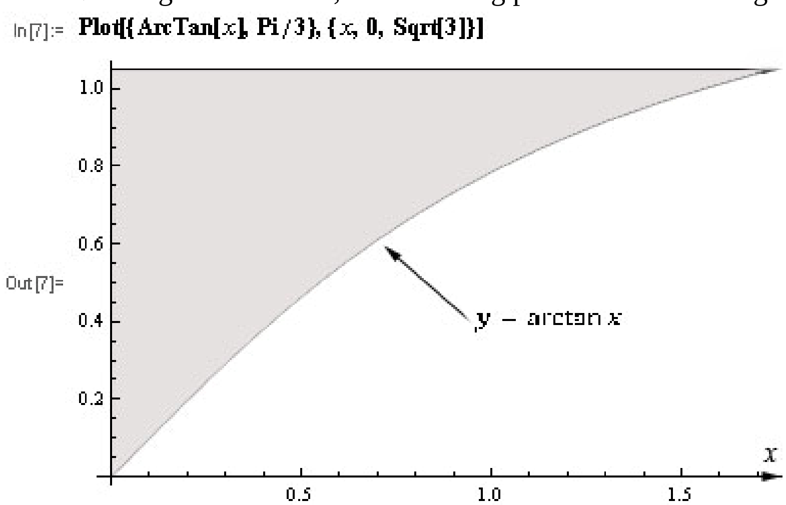
$$\int_0^5 \int_0^{25-5x} \int_{\frac{2x}{5}}^{\frac{25-3x-z}{5}} dy\,dz\,dx = \int_0^5 \int_0^{25-5x} \frac{25-5x-z}{5} dz\,dx = \frac{1}{5}\int_0^5 (25-5x)z - \frac{z^2}{2}\Big|_0^{25-5x} dx$$

$$= \frac{1}{5}\int_0^5 (25-5x)^2 - \frac{(25-5x)^2}{2} dx = \frac{1}{10}\int_0^5 (25-5x)^2 dx$$

$$= -\frac{1}{150}(25-5x)^3\big|_0^5 = 0 + \frac{625}{6}$$

2. Sketch the region of integration for $\int_0^{\sqrt{3}} \int_{\arctan x}^{\pi/3} f(x, y)\,dydx$ and change the order of integration.

SOLN: Using Mathematica, the following plot illustrates the region of integration:




SOLN: $y = \arctan x \Leftrightarrow x = \tan y$, so switching the bounds of integration yields $\int_0^{\pi/3} \int_0^{\tan y} f(x, y)\,dxdy$

3. Define the improper integral

where $S_a$ is the square with vertices and .

Use this to show that

SOLN: Substitute so that

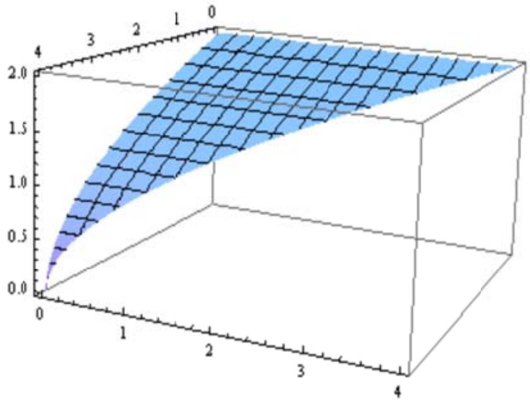
. Since the integrand is separable,

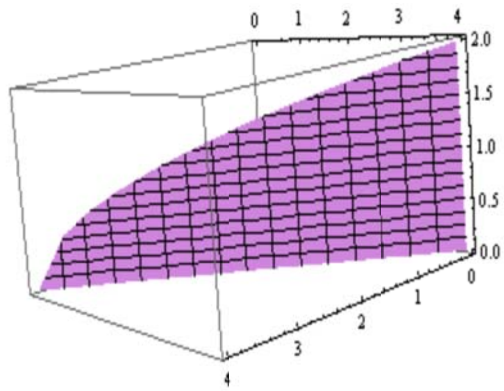
, which implies that

4. Evaluate is bounded by the cylinder , and the planes $x + y = 4$, $x = 0$, $y = 0$, and $z = 0$.

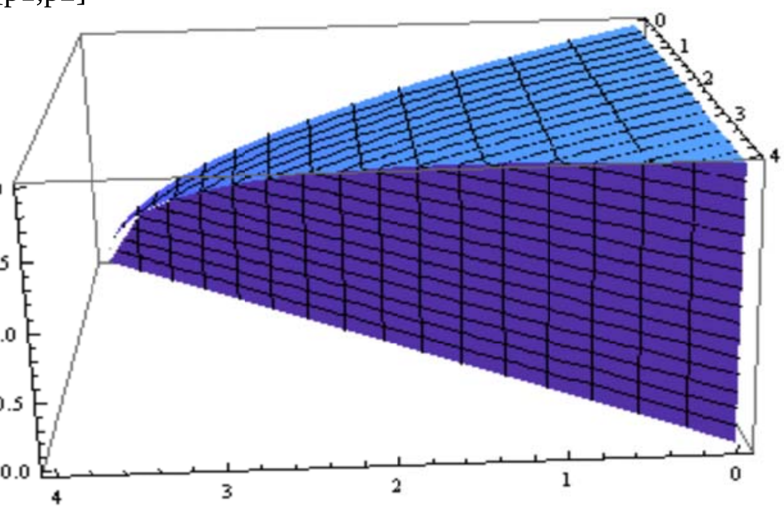
SOLN: We can visualize the boundary of this region by using ParametricPlot3D in Mathematica:

p1=ParametricPlot3D[{4-u^2,v,u},{u,0,2},{v,0,u^2}]     p2=ParametricPlot3D[{u,4-u,v},{u,0,4},{v,0,Sqrt[4-u]}]



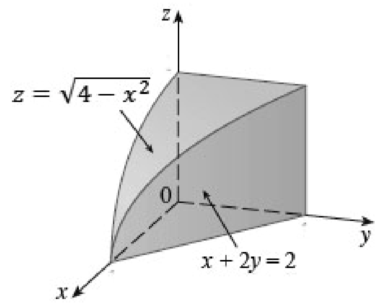




Show[p1,p2]




An integral for computing the integral is

5. Consider the density $\rho(x,y,z) = 1 + x^2 + y^2 + z^2$ over the volume bounded by $x^2 + z^2 = 4$, $x + 2y = 2$, $z = 0$, $x = 0$ and $y = 0$, as shown at right. Set up but ***do not evaluate*** three iterated integrals to compute moments about the




a. *xy*-plane.

$$M_{xy} = \int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{\frac{2-x}{2}} z(1 + x^2 + y^2 + z^2)\,dy\,dx\,dz$$

b. *xz*-plane.

$$M_{xz} = \int_0^1 \int_0^{2-2y} \int_0^{\sqrt{(4-x^2)}} x(1 + x^2 + y^2 + z^2)\,dz\,dy\,dy$$

c. $yz -$ plane.

$$M_{yz} = \int_0^2 \int_0^{4-x} \int_0^{\sqrt{(4-x^2)}} x(1 + x^2 + y^2 + z^2)\,dz\,dy\,dx$$

6. Use spherical coordinates to find the volume of the part of a sphere of $\rho = cos\phi$ that lies between the cones $\phi = \pi/4$ and $\phi = \pi/3$.

SOLN: $\int_0^{2\pi} \int_{\pi/4}^{\pi/3} \int_0^{\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta = \frac{2\pi}{3} \int_{\pi/4}^{\pi/3} \cos^3\phi \sin\phi\, d\phi = -\frac{\pi}{6} \cos^4\phi \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = -\frac{\pi}{6}\left(\frac{1}{16} - \frac{1}{4}\right) = \frac{\pi}{32}$

7. Determine the value of $\iint_D \sqrt{\frac{x+y}{x-2y}}\, dA$ where $D$ is the region in $\mathbb{R}^2$ enclosed by the lines $y = \frac{x}{2}$, $y = 0$, and $x + y = 1$. Hint: make a convenient change of variables.

SOLN: Let $u = x + y$ and $v = x - 2y$ so that $3x = 2u + v$ and $3y = u - v$, whence $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = \left|-\frac{2}{9} - \frac{1}{9}\right| = \frac{1}{3}$ and the boundary $y = \frac{x}{2}$ becomes $v = 0$, $y = 0$ is transformed to $v = 0$ and $x + y = 1$ to $u = 1$, so that $\iint_D \sqrt{\frac{x+y}{x-2y}}\, dA = \frac{1}{3} \int_0^1 \int_0^u \sqrt{\frac{u}{v}}\, dv\, du = \frac{2}{3} \int_0^1 \sqrt{uv}\Big|_0^u du = \frac{1}{3}$