Math 2A – Stewart, Early Transcendentals, 6e, 7.4#55

A function is called **homogeneous of degree *n*** if it satisfies the equation $f(tx,ty)=t^n f(x,y)$ for all *t*, where *n* is a positive integer and *f* has continuous second order partial derivatives.

(a) $f(x,y) = x^2y + 2xy^2 + 5y^3$ is homogeneous. This is easy.

(b) Show that if *f* is homogeneous of degree *n*, then $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y)$

SOLN: Let $g(t)= f(tx,ty)-t^n f(x,y)=0$.

Then $g'(t)=xf_x(tx,ty)+yf_y(tx,ty)-nt^{n-1}f(x,y)=0$

Since this must be true for all *t*. $g'(1)=xf_x(x,y)+yf_y(x,y)-nf(x,y)=0$ and bingo!

Follow-up: Note that $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\overrightarrow{\nabla f}\cdot\langle x,y\rangle=nf(x,y)$. In higher dimensions with the domain vector $\vec{x}=\langle x_1,x_2,\ldots,x_m\rangle=nf(\vec{x})$, this generalizes to $\overrightarrow{\nabla f}\cdot\vec{x}=nf(\vec{x})$ and the proof above works just fine, but the result is actually stronger: a theorem known as Euler's theorem for homogeneous functions, which states that if $f:\mathbb{R}_+^m\to\mathbb{R}$ is differentiable then it is homogeneous if and only if $\overrightarrow{\nabla f}\cdot\vec{x}=nf(\vec{x})$. We've proved that if it's homogeneous then $\overrightarrow{\nabla f}\cdot\vec{x}=nf(\vec{x})$. Can you prove the reverse? That is, if $\overrightarrow{\nabla f}\cdot\vec{x}=nf(\vec{x})$ then *f* is homogeneous?