Math 2A – Chapter 12 Problems – Fall '11 Name___________________________

Instructions: Write all responses on separate paper.
Show your work for credit.
Do not use a calculator.

1. Consider the force vector $\vec{F} = 5\hat{i} + 12\hat{j}$ and distance vector $\vec{d} = \hat{i} + \hat{j} + \hat{k}$.

   a. Find a unit vector (length = 1) in the direction of $\vec{d}$ and write this component-wise in positional (brocket) form: $\langle \_,\_,\_ \rangle$.

   b. Compute $\sqrt{\vec{F}\cdot\vec{F}}$, $\sqrt{\vec{d}\cdot\vec{d}}$ and $\vec{F}\cdot\vec{d}$.

   c. Express the angle between $\vec{F}$ and $\vec{d}$ as a simplified expression involving the arccos() function.

   d. Find the component of $\vec{F}$ in the direction of $\vec{d}$ = $\text{comp}_{\vec{d}}\vec{F}$ (note: this is a scalar.)

   e. Find projection of $\vec{F}$ onto $\vec{d}$ = $\text{proj}_{\vec{d}}\vec{F}$ (note: this is a vector.)

   f. Give a physical interpretation of $\vec{F}\cdot\vec{d}$.

   g. Compute $\vec{F}\times\vec{d}$.

   h. Compute $\left|\vec{F}\times\vec{d}\right|$.

   i. Give a geometric interpretation of $\left|\vec{F}\times\vec{d}\right|$.

   j. Give a physical interpretation of $\vec{F}\times\vec{d}$.

2. Consider the plane $z = 3 - 2x + y$.

   a. Complete the table below to find the three coordinate intercepts of the plane:

x	y	z
	0	0
0		0
0	0	

   b. Let *P* be the *x*-intercept, *Q* the *y*-intercept and *R* the *z*-intercept. Compute $\overrightarrow{PQ}\times\overrightarrow{PR}$.

c. Find the area of the triangle formed by *P*, *Q* and *R*.

3. Find an equation for the plane through (0,0,1) and perpendicular to the line $r = \langle 17,18,19 \rangle + t\langle 1,2,3 \rangle$.

4. Consider the two planes described by $2x - 4y - 8z + 8 = 0$ and $3x - 6y + 9z - 18 = 0$.

   a. Find the point *P* where these two planes intersect in the plane $x = 0$.

   b. Find the point *Q* where these two planes intersect in the plane $y = 0$.

   c. Use the points *P* and *Q* to find the vector form, $\vec{r} = \vec{r}_0 + t\vec{v}$, for the equation of the line where these two planes intersect.

5. Identify the surface whose equation is given as one of the following:

   (i) hyperbolic paraboloid,
   (ii) elliptical cone,
   (iii) elliptical paraboloid,
   (iv) ellipsoid
   (v) hyperboloid of one sheet
   (vi) hyperboloid of two sheets.
   (vii) hyperbolic cylinder

   and sketch a rough graph for each showing some traces in perspective (do your best)

   a. $x^2 - y^2 = 4$

   b. $x^2 - 2y + 3z^2 = 12$

   c. $x^2 + 2x + y^2 + z^2 = 0$ hint: first complete the square for *x*.

Math 2A – Chapter 12 Problems Solutions – Fall '11

1. Consider the force vector $\vec{F} = 5\hat{\imath} + 12\hat{\jmath}$ and distance vector $\vec{d} = \hat{\imath} + \hat{\jmath} + \hat{k}$.

a. Find a unit vector (length = 1) in the direction of $\vec{d}$ and write this component-wise in positional (brocket) form: $\langle \_,\_,\_ \rangle$.

SOLN: $\vec{d} = \hat{\imath} + \hat{\jmath} + \hat{k} = \langle 1,1,1 \rangle$ so a unit vector in the direction of $\vec{d}$ is

$$\frac{\vec{d}}{|\vec{d}|} = \frac{\langle 1,1,1 \rangle}{\sqrt{1^2+1^2+1^2}} = \frac{\langle 1,1,1 \rangle}{\sqrt{3}} = \left\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right\rangle$$

b. Compute $\sqrt{\vec{F}\cdot\vec{F}}$, $\sqrt{\vec{d}\cdot\vec{d}}$ and $\vec{F}\cdot\vec{d}$.

SOLN: $\sqrt{\vec{F}\cdot\vec{F}} = \sqrt{5^2+12^2} = \sqrt{169} = 13$, $\sqrt{\vec{d}\cdot\vec{d}} = \sqrt{3}$ (see above) and $\vec{F}\cdot\vec{d} = \langle 5,12,0 \rangle \langle 1,1,1 \rangle = 5+12+0 = 17$

c. Express the angle between $\vec{F}$ and $\vec{d}$ as a simplified expression involving the arccos() function.

SOLN: $\vec{F}\cdot\vec{d} = |\vec{F}||\vec{d}|\cos\theta \Leftrightarrow \theta = \cos^{-1}\left(\frac{\vec{F}\cdot\vec{d}}{|\vec{F}||\vec{d}|}\right) = \cos^{-1}\left(\frac{17}{13\sqrt{3}}\right) = \cos^{-1}\left(\frac{17\sqrt{3}}{39}\right)$

d. Find the component of $\vec{F}$ in the direction of $\vec{d}$ = $\text{comp}_{\vec{d}}\vec{F}$ (note: this is a scalar.)

SOLN: $\text{comp}_{\vec{d}}\vec{F} = |\vec{F}|\cos\theta = \frac{\vec{F}\cdot\vec{d}}{|\vec{d}|} = \frac{17}{\sqrt{3}} = \frac{17\sqrt{3}}{3}$

e. Find projection of $\vec{F}$ onto $\vec{d}$ = $\text{proj}_{\vec{d}}\vec{F}$ (note: this is a vector.)

SOLN: $\text{proj}_{\vec{d}}\vec{F} = (\text{comp}_{\vec{d}}\vec{F})\frac{\vec{d}}{|\vec{d}|} = \left(\frac{17\sqrt{3}}{3}\right)\left\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right\rangle = \left\langle \frac{17}{3}, \frac{17}{3}, \frac{17}{3} \right\rangle$

f. Give a physical interpretation of $\vec{F}\cdot\vec{d}$.

SOLN: If a constant force $\vec{F}$ acts on a body, thereby producing a displacement of magnitude $|\vec{d}|$ along a straight line in the direction of $\vec{d}$ then the energy involved in this movement is $\vec{F}\cdot\vec{d}$.

g. Compute $\vec{F}\times\vec{d}$.

SOLN: $\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & 12 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 12\hat{\imath} - 5\hat{\jmath} - 7\hat{k} = \langle 12,-5,-7 \rangle$

h. Compute $\left|\vec{F}\times\vec{d}\right|$. SOLN: $\left|\vec{F}\times\vec{d}\right| = \left|\vec{F}\right|\left|\vec{d}\right|\sin\theta = \sqrt{12^2+(-5)^2+(-7)^2} = \sqrt{218}$

i. Give a geometric interpretation of $\left|\vec{F}\times\vec{d}\right|$. SOLN: The area of the triangle formed by $\vec{F}$ and $\vec{d}$.

j. Give a physical interpretation of $\vec{F}\times\vec{d}$. SOLN: This is the vector opposite of the usual right-handed torque, you could think of it as a clockwise, or left-handed torque. It measures how much the force $\vec{F}$ when acting on the terminal point of the vector $\vec{d}$, causes the initial point (the pivot point) to rotate in the clockwise direction. The convention is that the counterclockwise direction is positive, so this is the opposite of that.

2. Consider the plane $z = 3 - 2x + y$.

a. Complete the table below to find the three coordinate intercepts of the plane: SOLN:

x	y	z
3/2	0	0
0	–3	0
0	0	3

b. Let *P* be the *x*-intercept, *Q* the *y*-intercept and *R* the *z*-intercept. Compute $\overrightarrow{PQ}\times\overrightarrow{PR}$.

SOLN: $\overrightarrow{PQ} = \left\langle 0-\frac{3}{2}, -3-0, 0-0\right\rangle = \left\langle -\frac{3}{2}, -3, 0\right\rangle$ and $\overrightarrow{PR} = \left\langle 0-\frac{3}{2}, 0-0, 3-0\right\rangle = \left\langle -\frac{3}{2}, 0, 3\right\rangle$

Thus $\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{3}{2} & -3 & 0 \\ -\frac{3}{2} & 0 & 3 \end{vmatrix} = \left\langle -9, \frac{9}{2}, -\frac{9}{2}\right\rangle$

c. Find the area of the triangle formed by *P*, *Q* and *R*.

SOLN: The area of this triangle is half the area of the of the parallelogram formed by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ which can be computed as half the magnitude of the cross product:

$$A = \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \frac{1}{2}\sqrt{9^2+\left(\frac{9}{2}\right)^2+\left(\frac{9}{2}\right)^2} = \frac{1}{2}\sqrt{81+\frac{81}{2}} = \frac{1}{2}\sqrt{\frac{243}{2}} = \frac{9\sqrt{6}}{4}$$

3. Find an equation for the plane through (0,0,1) and perpendicular to the line $r = \langle 17,18,19\rangle + t\langle 1,2,3\rangle$.

SOLN: The vector $\vec{n} = \langle 1,2,3\rangle$ is perpendicular to the plane, so if $\vec{P} = \langle x,y,z\rangle$ is an arbitrary vector to the plane then the vector $\overrightarrow{P_0P} = \langle x, y, z-1 \rangle$ is parallel to the plane and so $\overrightarrow{P_0P} \cdot \vec{n} = \langle x, y, z-1 \rangle \cdot \langle 1, 2, 3 \rangle = \boxed{x + 2y + 3z - 3 = 0}$ gives an equation for the plane.

4. Consider the two planes described by $2x - 4y - 8z + 8 = 0$ and $3x - 6y + 9z - 18 = 0$.

a. Find the point P where these two planes intersect in the plane $x = 0$.

SOLN: This is done by substituting $x = 0$ into the equations and solving the 2x2 system:

$\left(\begin{array}{cc|c} 4 & 8 & 8 \\ -6 & 9 & 18 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 4 & 4 \\ -2 & 3 & 6 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 7 & 10 \end{array}\right)$ so $z = 10/7$ and $y = 2 - 20/7 = -6/7$

b. Find the point Q where these two planes intersect in the plane $y = 0$.

SOLN: $\left(\begin{array}{cc|c} 2 & -8 & -8 \\ 3 & 9 & 18 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 4 & 4 \\ 1 & 3 & 6 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -4 & -4 \\ 0 & 7 & 10 \end{array}\right)$ so $z = 10/7$ and $x = -4 + 40/7 = 12/7$

c. Use the points P and Q to find the vector form, $\vec{r} = \vec{r}_0 + t\vec{v}$, for the equation of the line where these two planes intersect.

SOLN: The *yz*-plane intercept is $P_1 = \left(0, -\frac{6}{7}, \frac{10}{7}\right)$ and the *xz*-plane intercept is $P_2 = \left(\frac{12}{7}, 0, \frac{10}{7}\right)$ so an equation for the line $\vec{r} = \left\langle 0, -\frac{6}{7}, \frac{10}{7} \right\rangle + t\left\langle \frac{12}{7}, \frac{6}{7}, 0 \right\rangle$

5. Identify the surface whose equation is given as one of the following:

(i) hyperbolic paraboloid,
(ii) elliptical cone,
(iii) elliptical paraboloid,
(iv) ellipsoid
(v) hyperboloid of one sheet
(vi) hyperboloid of two sheets.
(vii) hyperbolic cylinder

and sketch a rough graph for each showing some traces in perspective (do your best)

a. $x^2 - y^2 = 4$

SOLN: The solution in the *xy*-plane is a hyperbola, so in R^3, it is a hyperbolic cylinder, part of which can be plotted in Mathematica with the command,

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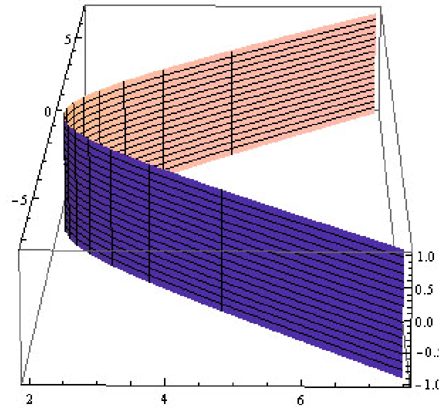
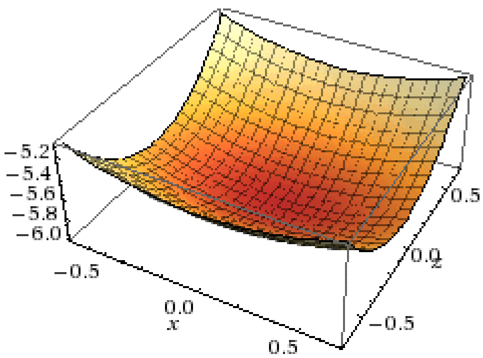
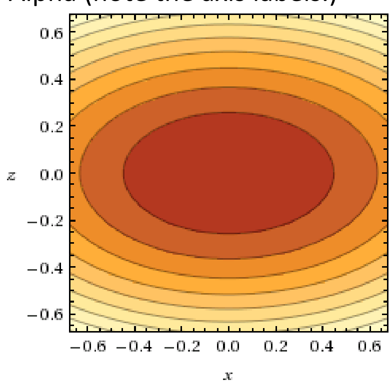
ParametricPlot3D [{2 * Sec[u],2 * Tan[u],v},{u,-1.3,1.3},{v,-
```

and looks roughly as shown at right:



b. $x^2 - 2y + 3z^2 = 12$ SOLN: $x^2 - 2y + 3z^2 = 12 \Leftrightarrow y = \frac{1}{2}x^2 + \frac{3}{2}z^2 - 6$ is a elliptical paraboloid opening outwards from a vertex at (0, –6 ,0). Wolfram Alpha (note the axis labels.)





c. $x^2 + 2x + y^2 + z^2 = 0$

hint: first complete the square for *x*.

SOLN:

$$x^2 + 2x + y^2 + z^2 = 0 \Leftrightarrow (x+1)^2 + y^2 + z^2 = 1$$

is an equation for the sphere (ellipsoid) of radius 1 centered at (–1,0,0). Not the scaling and labeling of the axes in the Wolfram Alpha rendering at right .