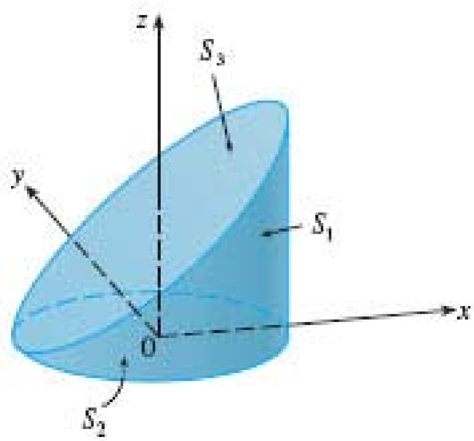
Math 2A – Chapter 16 Problems – Fall '11. Name________________________________
Show your work for credit. Write all responses on separate paper. Do not abuse a calculator.

1. Let $\vec{F} = \langle \sin y, x\cos y, \cos z \rangle$.
   a. Show that $\vec{F}$ is a gradient field
   b. Evaluate $\oint_C \vec{F} \cdot \overrightarrow{dr}$ where C is the curve $\vec{r}(t) = \langle \cos t, 2\sin t, \cos 2t \rangle$, $0 \le t \le 2\pi$.

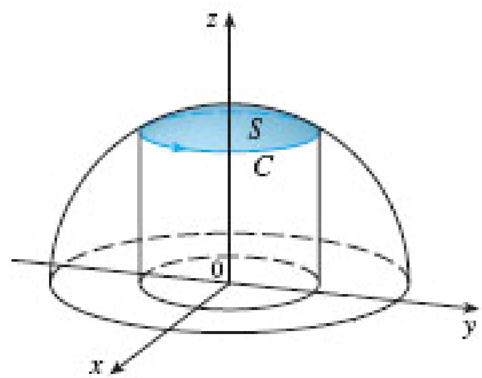
2. Evaluate $\oint_C \vec{F} \cdot \overrightarrow{dr}$ for $\vec{F} = \langle z-y, x-z, y-x \rangle$ and C: $\vec{r}(t) = \langle \cos t, \sin t, 1 \rangle$
   a. directly as a line integral.
   b. using Stokes' theorem.

3. Evaluate the surface integral $\iint_S \sqrt{1+x^2+y^2}\, dS$ where $S = S_1 + S_2 + S_3$ as shown in the diagram. Note that the surfaces can be parameterized as
   S_1: $\vec{r}(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle$, where $0 \le \theta \le 2\pi$ and $0 \le z \le 1 + \cos\theta$,
   S_2: $\vec{r}(R, \theta) = \langle R\cos\theta, R\sin\theta, 0 \rangle$, where $0 \le R \le 1$ and $0 \le \theta \le 2\pi$, and
   S_3: $\vec{r}(R, \theta) = \langle R\cos\theta, R\sin\theta, 1 + R\cos\theta \rangle$ where $0 \le R \le 1$ and $0 \le \theta \le 2\pi$.



4. Let $\vec{F} = \langle zx, yz, xy \rangle$.
   a. Compute the flux of $\vec{F}$ through the surface S of problem #3, above.

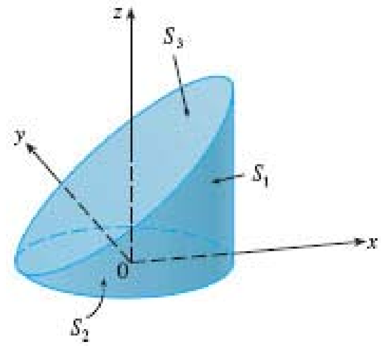
   b. Use Stokes' Theorem to compute $\iint_S \vec{\nabla} \times \vec{F} \cdot \overrightarrow{dS}$ where S is the part of the sphere $x^2 + y^2 + z^2 = 49$ that lies inside the cylinder $x^2 + y^2 = 36$ and above the xy-plane, as shown at right.



Math 2A – Chapter 16 Problem Solutions – Fall '11

1. Let $\vec{F} = \langle \sin y, x\cos y, \cos z\rangle$.
   a. Show that $\vec{F}$ is a gradient field
      SOLN: $\vec{F} = \overrightarrow{\nabla f}$ where $f(x,y,z) = x\sin y + \sin z$
   b. Evaluate $\oint_C \vec{F}\cdot\overrightarrow{dr}$ where C is the curve $\vec{r}(t) = \langle \cos t, 2\sin t, \cos 2t\rangle,\ \ 0 \le t \le 2\pi$.
      SOLN: The quick answer here is that this is zero because the fundamental theorem of line integrals says $\oint_C \vec{F}\cdot\overrightarrow{dr} = 0$ for any closed curve C in a conservative vector field. Working through the details we have that
      $\oint_C \vec{F}\cdot\overrightarrow{dr} = \int_0^{2\pi}\langle \sin(2\sin t), \cos t\cos(2\sin t), \cos(\cos 2t)\rangle \cdot \langle -\sin t, 2\cos t, -2\sin 2t\rangle dt\ =$
      $$\int_0^{2\pi} -\sin(2\sin t)\sin t + 2\cos^2 t\cos(2\sin t) - 2\cos(\cos 2t)\sin 2t\, dt$$
      Now it doesn't seem readily obvious how to evaluate this using the FTC. In general, we have
      $$\oint_C \overrightarrow{\nabla f}\cdot\overrightarrow{dr} = \oint_C \langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\rangle\cdot\langle dx, dy, dz\rangle = \int_a^b \frac{d}{dt}f(t)dt = f\big(\vec{r}(b)\big) - f\big(\vec{r}(a)\big)$$
      In this case that would be $f(1,0,1) - f(1,0,1) = 0$. Breaking it down to the individual components, say, $\int_a^b \frac{\partial f(x,y,z)}{\partial x}\frac{dx}{dt}dt\ = -\int_0^{2\pi}\sin y\sin t\, dt = -\int_0^{2\pi}\sin(2\sin t)\sin t\, dt$, substitute $u = \sin t$ so that $du = \cos t\, dt$ and the integral becomes $-\int_0^0 \frac{u\sin(2u)du}{\sqrt{1-u^2}} = 0$. Similarly, $\int_a^b \frac{\partial f(x,y,z)}{\partial y}\frac{dy}{dt}dt\ =$ $\int_0^{2\pi} 2x\cos y\ \cos t\, dt = \int_0^{2\pi} 2\cos^2 t\cos(2\sin t)\, dt$. Substituting $u = \sin t$ we have $du = \cos t\, dt$ whence this integral becomes $\int_1^1 2\sqrt{1-u^2}\sin 2u\, du = 0$.
2. Evaluate $\oint_C \vec{F}\cdot\overrightarrow{dr}$ for $\vec{F} = \langle z-y, x-z, y-x\rangle$ and C: $\vec{r}(t) = \langle \cos t\ , \sin t, 1\rangle$
   a. directly as a line integral.
      SOLN: $\oint_C \vec{F}\cdot\overrightarrow{dr} = \int_0^{2\pi}\langle 1-\sin t, \cos t - 1, \sin t - \cos t\rangle\cdot\langle -\sin t, \cos t, 0\rangle dt$
      $$= \int_0^{2\pi} -\sin t + \sin^2 t + \cos^2 t - \cos t\, dt = t + \cos t - \sin t\big|_0^{2\pi} = 2\pi$$
   b. using Stokes' theorem.
      SOLN: The simplest surface surrounded by this boundary is $\vec{r}(R,\theta) = \langle R\cos\theta, R\sin\theta, 1\rangle$ for which $\overrightarrow{dS} = (\vec{r}_R \times \vec{r}_\theta)dRd\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -R\sin\theta & R\cos\theta & 0\end{vmatrix} dRd\theta = \langle 0,0,R\rangle dRd\theta$
      We compute the curl by
      $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & x-z & y-x\end{vmatrix} = \langle 1-(-1), 1-(-1), 1-(-1)\rangle = \langle 2,2,2\rangle$ so that
      $$\oint_C \vec{F}\cdot\overrightarrow{dr} = \iint_D \vec{\nabla}\times\vec{F}\cdot\overrightarrow{dS} = \iint_D \langle 2,2,2\rangle\cdot\langle 0,0,R\rangle dRd\theta = \int_0^{2\pi}\int_0^1 2RdRd\theta = 2\pi$$

3. Evaluate the surface integral $\iint_S \sqrt{1+x^2+y^2}dS$ where $S = S_1 + S_2 + S_3$ as shown in the diagram. Note that the surfaces can be parameterized as
S_1: $\vec{r}(\theta, z) = \langle \cos\theta, \sin\theta, z\rangle$, where
$0 \le \theta \le 2\pi$ and $0 \le z \le 1 + \cos\theta$,
S_2: $\vec{r}(R, \theta) = \langle R\cos\theta, R\sin\theta, 0\rangle$, where
$0 \le R \le 1$ and $0 \le \theta \le 2\pi$, and
S_3: $\vec{r}(R, \theta) = \langle R\cos\theta, R\sin\theta, 1 + R\cos\theta\rangle$
where $0 \le R \le 1$ and $0 \le \theta \le 2\pi$.



SOLN: $\iint_S \sqrt{1+x^2+y^2}dS = +\iint_{S_1}\sqrt{1+x^2+y^2}dS + \iint_{S_2}\sqrt{1+x^2+y^2}dS + \iint_{S_3}\sqrt{1+x^2+y^2}dS$

For S_1: $\vec{r}(\theta, z) = \langle \cos\theta, \sin\theta, z\rangle$, $dS = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} d\theta dz = |\langle \cos\theta, \sin\theta, 0\rangle| d\theta dz = d\theta dz$

So $\iint_{S_1}\sqrt{1+x^2+y^2}dS = \int_0^{2\pi}\int_0^{1+\cos\theta}\sqrt{2}d\theta dz = \sqrt{2}\int_0^{2\pi} 1 + \cos\theta\, d\theta = 2\sqrt{2}\pi$

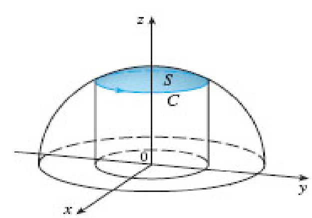
For S_2: $\vec{r}(R, \theta) = \langle R\cos\theta, R\sin\theta, 0\rangle$, $dS = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -R\sin\theta & R\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} d\theta dR = |\langle 0,0,-R\rangle| d\theta dR$

So $\iint_{S_2}\sqrt{1+x^2+y^2}dS = \int_0^{2\pi}\int_0^1 \sqrt{1+R^2}RdRd\theta = \pi\int_1^2\sqrt{u}du = \frac{2\pi}{3}u^{\frac{3}{2}}\Big|_1^2 = \frac{2\pi}{3}\left(2\sqrt{2}-1\right)$

For S_3: $\vec{r}(R, \theta) = \langle R\cos\theta, R\sin\theta, 1 + R\cos\theta\rangle$, $dS = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \sin\theta & \cos\theta \\ -R\sin\theta & R\cos\theta & -R\sin\theta \end{vmatrix} dRd\theta =$
$|\langle -R, 0, R\rangle| dRd\theta$ so $\iint_{S_3}\sqrt{1+x^2+y^2}dS = \sqrt{2}\int_0^{2\pi}\int_0^1\sqrt{1+R^2}RdRd\theta = \pi\left(2-\sqrt{2}\right)$

All together, $\iint_S \sqrt{1+x^2+y^2}dS = 2\sqrt{2}\pi + \frac{2\pi}{3}\left(2\sqrt{2}-1\right) + \pi\left(2-\sqrt{2}\right) = \frac{\pi}{3}\left(4+7\sqrt{2}\right)$

4. Let $\vec{F} = \langle zx, yz, xy\rangle$.
   a. Compute the flux of $\vec{F}$ through the surface S of problem #3, above.
      SOLN: According to Gauss' Theorem (The Divergence Theorem) $\oiint_S \vec{F}\cdot\overrightarrow{dS} = \iiint_E \vec{\nabla}\cdot\vec{F}dV$. Here $\vec{\nabla}\cdot\vec{F} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = 2z$ so
      flux = $\oiint_S \vec{F}\cdot\overrightarrow{dS} = \int_0^{2\pi}\int_0^1\int_0^{1+\cos\theta} 2zdz\, rdrd\theta = \int_0^{2\pi}(1+\cos\theta)^2 d\theta\int_0^1 rdr =$
      $$= \theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_0^{2\pi}\frac{1}{2} = \frac{3\pi}{2}$$
   b. Use Stokes' Theorem to compute
      $\iint_S \vec{\nabla}\times\vec{F}\cdot\overrightarrow{dS}$ where S is the part of the sphere $x^2+y^2+z^2 = 49$ that lies inside the cylinder $x^2+y^2 = 36$ and above the *xy*-plane, as shown at right.
      SOLN: The boundary of the surface is $\vec{r} = \langle 6\cos t, 6\sin t, \sqrt{13}\rangle$



$$\iint_S \vec{\nabla}\times\vec{F}\cdot\overrightarrow{dS} = \oint_C \vec{F}\cdot\overrightarrow{dr} = \int_0^{2\pi}\langle 6\sqrt{13}\cos t, 6\sqrt{13}\sin t, 36\cos t\sin t\rangle\langle -6\sin t, 6\cos t, 0\rangle dt$$
$=\int_0^{2\pi} 0dt = 0.$

On the other hand, the surface integral is computed by finding

$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zx & zy & xy \end{vmatrix} = \langle x - y, x - y, 0 \rangle$ and parameterizing the sphere as

$\vec{r}(\phi, \theta) = \langle 7 \sin\phi \cos\theta\,, 7 \sin\phi \sin\theta\,, 7\cos\phi \rangle$, we have

$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 7\cos\phi\cos\theta & 7\cos\phi\sin\theta & -7\sin\phi \\ -7\sin\phi\sin\theta & 7\sin\phi\cos\theta & 0 \end{vmatrix} =$

$\langle 49\sin^2\phi\cos\theta\,, 49\sin^2\phi\sin\theta\,, 49\cos\phi\sin\phi \rangle$ so that $\left|\vec{r}_\phi \times \vec{r}_\theta\right| = 49\sin\phi$.

$\overrightarrow{dS} = 49\langle \sin\phi\cos\theta\,, \sin\phi\sin\theta\,, \cos\phi \rangle \sin\phi\, d\phi d\theta$

And so (check this)

$\iint_S \vec{\nabla} \times \vec{F} \cdot \overrightarrow{dS} =$

$\int_0^{\arcsin\frac{6}{7}} \int_0^{2\pi} 343\sin\phi\,(\cos\theta - \sin\theta)\langle 1,1,0 \rangle \cdot \langle \sin\phi\cos\theta\,, \sin\phi\sin\theta\,, \cos\phi \rangle \sin\phi\, d\phi d\theta$

$$= 343 \int_0^{\arcsin\frac{6}{7}} \sin^3\phi d\phi \int_0^{2\pi} \cos 2\theta\, d\theta = 0$$