Math 12 – Precalculus – Take Home Problems for Chapter 7 Test
Write all responses on separate paper. Make your answers as detailed and clear as possible.

Use mathematical induction to prove equation 1, 2 and 3.

1. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
2. $\sum_{k=1}^{n} (2k-1)^3 = n^2(2n^2-1)$
3. $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

4. Show that the sequence of successive ratios F_{n+1}/F_n appears to converge to a number r satisfying the equation $r^2 = r + 1$. (The number r was known as the golden ratio to the ancient Greeks.)

5. Let r satisfy $r^2 = r + 1$. Show that the sequence $s_n = Ar^n$, where A is a constant, satisfies the Fibonacci equation $s_n = s_{n-1} + s_{n-2}$ for n > 2.

6. Find the term that does not contain x in the expansion of $\left(8x - \frac{1}{2x}\right)^8$.

7. Find the last three terms in the expansion $\left(a^{2/3} + a^{1/3}\right)^{25}$

8. Show that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

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1. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

   SOLN: For $n = 1$, $\sum_{k=1}^{1} k^2 = 1^2 = \frac{(1+1)(2+1)}{6} = \frac{6}{6}$.

   Assuming the statement is true at n, we have the next statement follows as shown:

   $$\sum_{k=1}^{n+1} k^2 = \left(\sum_{k=1}^{n} k^2\right) + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)\left[n(2n+1)+6(n+1)\right]}{6}$$

   $$= \frac{(n+1)\left[2n^2+7n+6\right]}{6} = \frac{(n+1)\left[(n+2)(2n+3)\right]}{6} = \frac{(n+1)\left[(n+1)+1\right]\left[2(n+1)+1\right]}{6}$$

2. $\sum_{k=1}^{n} (2k-1)^3 = n^2\left(2n^2-1\right)$

   SOLN: For $n = 1$, $\sum_{k=1}^{1} (2k-1)^3 = (2-1)^3 = 1^2\left(2(1)^2-1\right) = 1$

   Assuming the statement is true at n, we have the next statement follows as shown:

   $$\sum_{k=1}^{n+1} (2k-1)^3 = \left(\sum_{k=1}^{n} (2k-1)^3\right) + \left[2(n+1)-1\right]^3 = n^2\left(2n^2-1\right) + (2n+1)^3$$

   $$= 2n^4 - n^2 + 8n^3 + 12n^2 + 6n + 1 = \left(n^2+2n+1\right)\left(2n^2+4n+1\right) = (n+1)^2\left[2(n+1)^2-1\right]$$

3. $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

   SOLN: $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} = \frac{1}{1+1}$, so the the statement is true for $n = 1$.

   Assume $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$, then

   $$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \left(\sum_{k=1}^{n} \frac{1}{k(k+1)}\right) + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

   $$= \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}$$

4. Show that the sequence of successive ratios F_{n+1}/F_n appears to converge to a number r satisfying the equation $r^2 = r + 1$. (The number r was known as the golden ratio to the ancient Greeks.)

   SOLN: By definition of the Fibonacci numbers, $F_{n+1} = F_n + F_{n-1}$. Dividing through by F_n yields $\frac{F_{n+1}}{F_n} = 1 + \frac{1}{F_n / F_{n-1}}$. The key observation here is that if F_{n+1}/F_n converges to r as n gets large, then so does F_n / F_{n-1} converge to r and so for large n the equation becomes $r \approx 1 + 1/r$ or $r^2 \approx r + 1$ (in the limit, the approximation is exact.)

5. Let r satisfy $r^2 = r + 1$. Show that the sequence $s_n = Ar^n$, where A is a constant, satisfies the Fibonacci equation $s_n = s_{n-1} + s_{n-2}$ for n > 2.
   SOLN: We need to show that $s_n = s_{n-1} + s_{n-2} \Leftrightarrow Ar^n = Ar^{n-1} + Ar^{n-2}$ which follows immediately from dividing through by Ar^{n-2}.
6. Find the term that does not contain x in the expansion of $\left(8x - \frac{1}{2x}\right)^8$.

   SOLN: $\binom{8}{4}(8x)^4\left(-\frac{1}{2x}\right)^4 = \frac{8(7)(6)(5)}{4(3)(2)}\left(\frac{8}{2}\right)^4 = 70(256) = 17920$
7. Find the last three terms in the expansion $\left(a^{2/3} + a^{1/3}\right)^{25}$
   SOLN:
   $$\left(a^{2/3} + a^{1/3}\right)^{25} = a^{50/3} + \cdots + \binom{25}{2} a^{4/3} a^{23/3} + \binom{25}{1} a^{2/3} a^{24/3} + a^{25/3} = a^{50/3} + \cdots + 300a^9 + 25a^{26/3} + a^{25/3}$$

8. Show that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

   SOLN:
   $$\binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(n-r+1)!(r-1)!} + \frac{n!}{r!(n-r)!} = \frac{n!r + n!(n-r+1)}{r!(n-r+1)!}$$
   $$= \frac{n!(r + (n-r+1))}{r!(n-r+1)!} = \frac{n!(n+1)}{r!(n-r+1)!} = \binom{n+1}{r}$$