A King's Tour of the Chessboard

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The king in chess can move one square in any direction, horizontally, vertically, or diagonally. With no restrictions, a king's tour of the chessboard, visiting each square once and only once, is easy. (Conducting a knight's tour is a far more challenging puzzle that has received much study [**1**, pp. 257–66].) Now add the restriction that the king (after the first move) can only move to a square which touches an *even* number of squares which have already been visited. This problem is much harder; in fact, *it is impossible*.

The impossibility follows from a simple parity argument in the case of the standard 8×8 chessboard, or any chessboard with both dimensions even or both dimensions odd (except the trivial 1×1 board.) Two squares of the board are called a **neighbor pair** if they touch, either at their corners or along their sides. A neighbor pair is **completed** if both its squares have been visited by the king, including the case in which the king is still sitting on one of the two squares. The rules of the tour require the king to complete one neighbor pair on its first move and an even number of neighbor pairs on each subsequent move. Thus the total number of completed neighbor pairs (call it N) at any stage of the tour is odd. On a board with m rows and n columns, there are $m(n-1)$ horizontal neighbor pairs, $n(m-1)$ vertical neighbor pairs, and $2(m-1)(n-1)$ diagonal neighbor pairs. If the king can finish the tour, he will complete $N = 2mn - (m+n) + 2(m-1)(n-1)$ neighbor pairs. Since this total must be odd, $m+n$ must be odd, and thus m and n must have opposite parity.

In case the dimensions m, n of the board do not have the same parity, we need a different argument to prove a tour is impossible. We will keep track of another quantity that is not so directly related to the rules of the game. Four squares of the board that meet at a point (thus forming a 2×2 square) will be called a **foursome**. The foursomes are in one-to-one correspondence with internal vertices of an $m \times n$ board, and so there are $(m-1)(n-1)$ of them. A foursome is **completed** when the king has visited its four squares, including the case where the king is still sitting on the fourth square. We will let F stand for the number of foursomes that the king has completed at a given stage of the tour.

We shall use the letter S to denote the total number of squares that the king has visited. (This is one more than the number of moves the king has made.) In analyzing many tours, we discovered that the quantity

$$I = S + F - \frac{N+1}{2} \tag{1}$$

has the property that it cannot increase (with one exception) as the tour progresses.

Lemma. *Except for the first and last move, the value of I either decreases or stays the same.*

Proof. We will say that on a given move, the quantities I, S, F and N change by amounts ΔI, ΔS, ΔF and ΔN. The definition of a move gives us $\Delta S = 1$ and the "even rule" gives us $\Delta N = 2$, 4, 6 or 8 (the rules of chess eliminate $\Delta N = 0$). The geometry of the board gives us $\Delta F = 0, 1, 2, 3$ or 4 and the definition of I in (1) gives us $\Delta I = \Delta S + \Delta F - \Delta N/2 = 1 + \Delta F - \Delta N/2$.

We want to show that, except for the last move, $\Delta I \leqslant 0$. We do this by looking at the possible values of ΔF in turn. If $\Delta F = 0$, then $\Delta I \leqslant 0$ since $\Delta N \geqslant 2$. If $\Delta F = 1$, then the latest move completes at least three neighbor pairs, and so by the "even rule" $\Delta N \geqslant 4$ and $\Delta I \leqslant 0$. If $\Delta F = 2$ then the two completed foursomes overlap at least in the latest square visited, and at most in that square and one other square. In the former case, the king is adjacent to at least six filled squares, and in the latter case, to at least five filled squares. By the even rule, $\Delta N \geqslant 6$ in either case, and

FIGURE 1

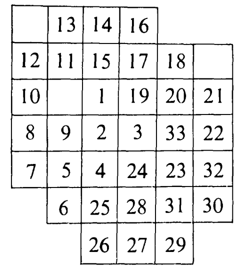

FIGURE 2

this makes $\Delta I \leqslant 0$. If $\Delta F \geqslant 3$, then $\Delta N \geqslant 7$, implying $\Delta N = 8$, $\Delta F = 4$ and $\Delta I = 1$. But after such a move, the king is sitting on a square that is surrounded by eight squares he has already visited, and so his tour is over.

The first two moves of a tour can only occur in the two equivalent ways shown in FIGURE 1. These create a configuration in which $I = 1$. A consequence of the Lemma is that no subsequent value of I can ever be larger than 2. Using the Lemma, we can complete the proof of our assertion made at the outset of this note.

THEOREM. *A king's tour under the stated conditions cannot be completed on any rectangular chessboard except a* 1×1 *or* 1×2.

Proof. Recall that on an $m \times n$ chessboard, $S = mn$, $F = (m-1)(n-1)$, $N = m(n-1) + n(m-1) + 2(m-1)(n-1)$, and so $I = (1/2)(m+n-1)$. If $m + n \geqslant 7$, then $I \geqslant 3$ and the tour cannot be completed. If $m + n = 5$, we have a 1×4 or 2×3 chessboard. Here, the value of I is 2, but it is not possible to make a move with $\Delta I = 1$ because, from the proof of the Lemma, such a move must complete eight neighbor pairs. But no square on the board has eight neighbors. If $m + n = 3$, we have a 1×2 board, on which the tour *can* be completed because the king is allowed to violate the even rule on the first move.

The Theorem leaves open the question of what fragments of chessboards *can* be toured. This problem, without the chess king metaphor, was originally proposed by Sid Sackson in [**2**] for the fragment shown in FIGURE 2. Since Sackson's board has $S = 36$, $N = 111$, $F = 24$ and hence $I = 4$, a tour of his board is impossible. The tour indicated in FIGURE 2, which visits 33 of the 36 squares, was found using a computer program written by Mark Meyerson. The method of this paper does not rule out a tour of 34 squares, omitting two isolated internal squares. A computer search indicates, however, that no such tour is possible.

Sackson also proposed the rule that each square visited must be adjacent to an *odd* number of squares already visited. He showed that his board can be toured using this rule, but our computer searches have turned up no instances of rectangular boards other than $1 \times n$ boards that can be toured. The quantity I in (1) rises and falls uncontrollably under these moves. (Necessarily: if the value of I only decreased under odd as well as even moves, then an unrestricted king's tour of the chessboard would be impossible, which is absurd.)

Note that we use the chess rule that a king must move to an adjacent square only when we argue that a move with $\Delta N = 8$ must end the tour and when we state that $\Delta N \neq 0$. Such moves are illegal using the "odd" rule; a complete king's tour of a rectangular chessboard must end on the edge of the board under this rule. This distinction may account for our inability thus far to come up with a quantity comparable to I for the odd rule.

Supported in part by grants from the Naval Academy Research Council.

References

[1] Maurice Kraitchik, Mathematical Recreations, 2nd ed., Dover, Inc., New York, 1953.

[2] S. Sackson, Odd & even, Games, 6 (1982) 53.