Math 2B – Spring ’13 – Chapters 5 and 6 Test Name______________________________

Give complete solutions to all problems. You can use technology to check your results, but be sure to detail how they are derived. Do not copy other people’s work.

1. Find the LU factorization of $A = \begin{bmatrix} 2 & 9 & 8 & 7 \\ 10 & 46.5 & 44 & 38 \\ 6 & 28.5 & \frac{88}{3} & 26 \\ 0 & 0 & \frac{20}{3} & 11.25 \end{bmatrix}$ and use it to compute the determinant, $|A^n|$ in terms of *n*.

2. Assuming we do not perform any row permutations, what is the 5th and last pivot when transforming the following matrix A into an upper triangular matrix by row eliminations? Explain.
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 2 & 4 & 6 & 0 \\ -1 & -2 & 2 & 4 & 0 \\ 5 & 4 & 3 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 \end{bmatrix}$$

3. Is there an LDL^T factorization of $A = \begin{bmatrix} 16 & 2 & 9 & 13 \\ 2 & 11 & 5 & 13 \\ 9 & 9 & 6 & 5 \\ 13 & 14 & 5 & 1 \end{bmatrix}$. Explain.

4. Find the Cholesky factorization of $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. Is A positive definite? Explain.

5. The implicit diagonalization $AS = S\Lambda$ occurs in eigenvalue problems, where Λ is the diagonal matrix of eigenvalues.
   a. Describe the significance of the individual columns of S in relation to the diagonal entries in Λ.
   b. Describe two uses of the diagonalization factorization.

6. Let $A = \begin{pmatrix} -3 & 2 & 4 \\ 2 & -6 & 2 \\ 4 & 2 & -3 \end{pmatrix}$
   a. Find the eigenvalues of A.
   b. Find an orthogonal matrix P such that $P^T AP$ is diagonal.
   c. Find an expression for $\exp(A)$.

7. (a) Argue that the eigenvalues of a permutation matrix must all have a modulus equal to 1.

   (b) What are the eigenvalues of $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$?

   (c) Express the eigenvalues in polar form.

8. If $AS = S\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 1/6 \end{bmatrix}$ and S is invertible.
   a. What is the determinant of A?

b. What are the eigenvalues of $A - I$? Of $A - A^{-1}$? And of A^{-1}?
c. Find a factored form of e^{tA} in terms of *S* and the diagonal matrix above.
d. Calculate the trace of A and then the trace of e^{tA}.

9. The quadratic $5x^2 + 8xy + 5y^2 = 1$ describes a tilted ellipse in the *xy*-plane.
a. Use eigenvalue methods to find the lengths of the major and minor axes.
b. Find the directions of the axes of this tilted ellipse in the *xy*-plane

10. One important use of SVD is in finding orthonormal bases for some or all of the four fundamental subspaces. Other uses include finding the rank of a matrix and solving least squares problems. You can extract subspace information by selecting appropriate columns from U or V. The code below shows Octave output for a specific “magic matrix.” Use the SVD output to find an orthonormal basis for the column space $C(A)$ and the nullspace $N(A)$. What is the rank of A?

```
A=magic(4) % matrix in question
A =
16 2 3 13
5 11 10 8
9 7 6 12
4 14 15 1
octave:> help svd
-- Loadable Function: S = svd (A)
-- Loadable Function: [U, S, V] = svd (A)
-- Loadable Function: [U, S, V] = svd (A, ECON)
Compute the singular value decomposition of A
A = U*S*V'
The function 'svd' normally returns only the vector of singular
values. When called with three return values, it computes U, S,
and V. For example,
svd (hilb (3))
returns
ans =
1.4083189
0.1223271
0.0026873
and
[u, s, v] = svd (hilb (3))
returns
u =
-0.82704 0.54745 0.12766
-0.45986 -0.52829 -0.71375
-0.32330 -0.64901 0.68867
s =
1.40832 0.00000 0.00000
0.00000 0.12233 0.00000
0.00000 0.00000 0.00269
v =
-0.82704 0.54745 0.12766
-0.45986 -0.52829 -0.71375
-0.32330 -0.64901 0.68867
If given a second argument, `svd' returns an economy-sized
decomposition, eliminating the unnecessary rows or columns of U or V.
```

11. Suppose that A is 5 by 3 and $A = U\Sigma V^T$ is its SVD, where

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   a. Find the eigenvalues of $A^T A$ using $A = U\Sigma V^T$.
   b. Remember properties of orthogonal matrices and evaluate the determinants, $|AA^T|$ and $|A^T A|$.

12. Write $(x_1 + x_2 + x_3 + x_4)^2 + (x_1 - x_2 - x_3)^2 = x^T A x$ in terms of a symmetric matrix A and vector $x^T = [x_1, x_2, x_3, x_4]$. What is the rank of A?

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1. Find the LU factorization of $A = \begin{bmatrix} 2 & 9 & 8 & 7 \\ 10 & 46.5 & 44 & 38 \\ 6 & 28.5 & \frac{88}{3} & 26 \\ 0 & 0 & \frac{20}{3} & 11.25 \end{bmatrix}$ and use it to compute the determinant, $|A^n|$ in terms of *n*.

   SOLN:

$$\begin{bmatrix} 2 & 9 & 8 & 7 \\ 10 & 46.5 & 44 & 38 \\ 6 & 28.5 & \frac{88}{3} & 26 \\ 0 & 0 & \frac{20}{3} & 11.25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 & 8 & 7 \\ 0 & 1.5 & 4 & 3 \\ 0 & 1.5 & \frac{16}{3} & 5 \\ 0 & 0 & \frac{20}{3} & 11.25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 & 8 & 7 \\ 0 & 1.5 & 4 & 3 \\ 0 & 1.5 & \frac{16}{3} & 5 \\ 0 & 0 & \frac{20}{3} & 11.25 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 & 8 & 7 \\ 0 & 1.5 & 4 & 3 \\ 0 & 0 & \frac{4}{3} & 2 \\ 0 & 0 & \frac{20}{3} & 11.25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 & 8 & 7 \\ 0 & 1.5 & 4 & 3 \\ 0 & 0 & \frac{4}{3} & 2 \\ 0 & 0 & 0 & 1.25 \end{bmatrix} = LU$$

   The determinant of the product is the product of determinants. The determinant of *L* is 1, while the determinant of *U* is 5. Thus $|A| = 5$ and $|A^n| = 5^n$.

2. Assuming we do not perform any row permutations, what is the 5th and last pivot when transforming the following matrix *A* into an upper triangular matrix by row eliminations? Explain.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 2 & 4 & 6 & 0 \\ -1 & -2 & 2 & 4 & 0 \\ 5 & 4 & 3 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 \end{bmatrix}$$

   SOLN:

   The fifth pivot is given by $\frac{|A|}{C_{5,5}} = \frac{2C_{5,5}}{C_{5,5}} = 2$

3. Is there an LDL^T factorization of $A = \begin{bmatrix} 16 & 2 & 9 & 13 \\ 2 & 11 & 5 & 13 \\ 9 & 9 & 6 & 5 \\ 13 & 14 & 5 & 1 \end{bmatrix}$. Explain.

   SOLN: No, *A* is not symmetric.

4. Find the Cholesky factorization of $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. Is *A* positive definite? Explain.

   SOLN:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 1 & 3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = LDL^T$$

The Cholesky factorization is then $\left(L\sqrt{D}\right)\left(L\sqrt{D}\right)^T$ where $L\sqrt{D} = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{6}/2 & 0 & 0 \\ 0 & \sqrt{6}/3 & 2\sqrt{3}/3 & 0 \\ 0 & 0 & \sqrt{3}/2 & \sqrt{5}/2 \end{bmatrix}$

or, as Mathematica has it, $\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & \frac{2}{\sqrt{3}} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{5}}{2} \end{bmatrix}^T$. Yes, A is positive definite since the pivots have the same signs as the eigenvalues and the pivots are all positive.

5. The implicit diagonalization $AS = S\Lambda$ occurs in eigenvalue problems, where Λ is the diagonal matrix of eigenvalues.
   a. Describe the significance of the individual columns of S in relation to the diagonal entries in Λ.
      SOLN: The nth column of S is the eigenvector for the eigenvalue at $\Lambda_{n,n}$.
   b. Describe two uses of the diagonalization factorization.
      SOLN: (1) It's much easier to compute powers of a matrix if it's in diagonalized form.
      (2) It's easier to compute the determinant. (3) Get answer 5b right on the exam

6. Let $A = \begin{pmatrix} -3 & 2 & 4 \\ 2 & -6 & 2 \\ 4 & 2 & -3 \end{pmatrix}$
   a. Find the eigenvalues of A.

$$|A - \lambda I| = -(\lambda+3)^2(\lambda+6) + 32 + 16(\lambda+6) + 8(\lambda+3) = -\lambda^3 - 12\lambda^2 - 21\lambda + 98 = 0$$

Factoring, we find $(\lambda+7)^2(\lambda-2) = 0$ the eigenvalues are 2 and the repeated -7.

   b. Find an orthogonal matrix P such that P^TAP is diagonal.

The eigenvector for 2 spans the null space of $A - 2I = \begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

Now for -7 the situation is different. $A + 7I = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow N = \text{span}\left\{\begin{matrix} -1 & -1 \\ 2 & 0 \\ 0 & 1 \end{matrix}\right\}$

Trouble is, of course, that while these vectors span the space, they're not orthonormal. Gram Schmidt!

Take $A = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Then $B = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - \left(\frac{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}^T \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}^T \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}\right)\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$. Ha! They're already perpendicular!

So $C = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}^T \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}^T \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}\right)\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4/5 \\ -2/5 \\ 1 \end{bmatrix}$ Normalizing, we get the matrix

$$P^T = \begin{bmatrix} 2/3 & -\sqrt{5}/5 & -4\sqrt{5}/15 \\ 1/3 & 2\sqrt{5}/5 & -2\sqrt{5}/15 \\ 2/3 & 0 & \sqrt{5}/3 \end{bmatrix}$$

Thus $\begin{bmatrix} -3 & 2 & 4 \\ 2 & -6 & 2 \\ 4 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 2/3 & -\sqrt{5}/5 & -4\sqrt{5}/15 \\ 1/3 & 2\sqrt{5}/5 & -2\sqrt{5}/15 \\ 2/3 & 0 & \sqrt{5}/3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -\sqrt{5}/5 & 2\sqrt{5}/5 & 0 \\ -4\sqrt{5}/15 & -2\sqrt{5}/15 & \sqrt{5}/3 \end{bmatrix}$

…and that's a fact.

c. Find an expression for $\exp(A)$.

SOLN:

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \begin{bmatrix} 2/3 & -\sqrt{5}/5 & -4\sqrt{5}/15 \\ 1/3 & 2\sqrt{5}/5 & -2\sqrt{5}/15 \\ 2/3 & 0 & \sqrt{5}/3 \end{bmatrix} \sum_{n=0}^{\infty} \begin{bmatrix} \frac{2^n}{n!} & 0 & 0 \\ 0 & \frac{(-7)^n}{n!} & 0 \\ 0 & 0 & \frac{(-7)^n}{n!} \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -\sqrt{5}/5 & 2\sqrt{5}/5 & 0 \\ -4\sqrt{5}/15 & -2\sqrt{5}/15 & \sqrt{5}/3 \end{bmatrix} =$$

$$\begin{bmatrix} 2/3 & -\sqrt{5}/5 & -4\sqrt{5}/15 \\ 1/3 & 2\sqrt{5}/5 & -2\sqrt{5}/15 \\ 2/3 & 0 & \sqrt{5}/3 \end{bmatrix} \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^{-7} & 0 \\ 0 & 0 & e^{-7} \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -\sqrt{5}/5 & 2\sqrt{5}/5 & 0 \\ -4\sqrt{5}/15 & -2\sqrt{5}/15 & \sqrt{5}/3 \end{bmatrix}$$

7. (a) Argue that the eigenvalues of a permutation matrix must all have a modulus equal to 1.

SOLN: Assume that λ is an eigenvalue of a permutation matrix P and v is the corresponding eigenvector so that $Pv = \lambda v$. Both v and λ may be complex (and often are). Now look at the sum of the moduli of the v_i's, i.e. $\sum_{i=0}^{n} |v_i|$. As P is a permutation matrix, Pv has the same entries as v except that they are in a different order. Thus

$$\sum_{i=1}^{n} |Pv_i| = \sum_{i=1}^{n} |v_i| = \sum_{i=1}^{n} |\lambda_i v_i| = \sum_{i=1}^{n} |\lambda_i||v_i|$$

That sure makes it seem as though the eigenvalues have modulus 1.

To gather more evidence in support of this, note the diagonalization of

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$ which is nicely interpreted as "first rotate clockwise by 45°, then reflect across the *x*-axis, then rotate counterclockwise by 45°. But the values are all real. How about $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$? Here the characteristic equation is $1-\lambda^3 = 0$ so the eigenvalues are cube roots of unity: $\lambda = 1, \frac{-1 \pm i\sqrt{3}}{2}$. For $\lambda = 1$, the eigenvector is $\frac{\sqrt{3}}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. For $\lambda = \frac{-1+i\sqrt{3}}{2}$ the eigenvector is $\frac{\sqrt{3}}{3}\begin{bmatrix} \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \\ 1 \end{bmatrix}$ and for $\lambda = \frac{-1-i\sqrt{3}}{2}$ the eigenvector is $\frac{\sqrt{3}}{3}\begin{bmatrix} \frac{-1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \\ 1 \end{bmatrix}$ Thus

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{-1+i\sqrt{3}}{2} & \frac{-1-i\sqrt{3}}{2} \\ 1 & \frac{-1-i\sqrt{3}}{2} & \frac{-1+i\sqrt{3}}{2} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{-1-i\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ \frac{-1+i\sqrt{3}}{2} & \frac{-1-i\sqrt{3}}{2} & 1 \\ \frac{-1-i\sqrt{3}}{2} & \frac{-1+i\sqrt{3}}{2} & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} =$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2}(-1 - i\sqrt{3}) & \frac{1}{2}(-1 + i\sqrt{3}) \\ 1 & \frac{1}{2}(-1 + i\sqrt{3}) & \frac{1}{2}(-1 - i\sqrt{3}) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1 + i\sqrt{3}) & 0 \\ 0 & 0 & \frac{1}{2}(-1 - i\sqrt{3}) \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2}(-1 - i\sqrt{3}) & \frac{1}{2}(-1 + i\sqrt{3}) \\ 1 & \frac{1}{2}(-1 + i\sqrt{3}) & \frac{1}{2}(-1 - i\sqrt{3}) \end{pmatrix}$$

Note that a non symmetric matrix has been factored as a product of symmetric matrices. These are the strange doings of complex valued matrices…

Let's push on to 4 × 4 before we go to our 5 × 5 problem.

$\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}$ has the characteristic equation $1-\lambda^4=0$ whose roots are $\lambda=\pm 1,\pm i$. I'm not going to type out all the details here, but it's easy to verify these.

$$\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}=\tfrac{1}{4}\begin{bmatrix}-1&i&-i&1\\1&-1&-1&1\\-1&-i&i&1\\1&1&1&1\end{bmatrix}\begin{bmatrix}-1&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}-1&1&0&0\\i&-1&1&0\\-i&0&0&1\\1&0&0&0\end{bmatrix}=$$

$$\frac{1}{4}\begin{pmatrix}-1&1&-1&1\\i&-1&-i&1\\-i&-1&i&1\\1&1&1&1\end{pmatrix}^T\begin{pmatrix}-1&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}-1&1&-1&1\\i&-1&-i&1\\-i&-1&i&1\\1&1&1&1\end{pmatrix}$$

(b) What are the eigenvalues of $\begin{bmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\end{bmatrix}$?

SOLN: The characteristic equation is $|A-\lambda I|=\begin{vmatrix}-\lambda&1&0&0&0\\1&-\lambda&0&0&0\\0&0&-\lambda&1&0\\0&0&0&-\lambda&1\\0&0&1&0&-\lambda\end{vmatrix}=-\lambda\begin{vmatrix}-\lambda&0&0&0\\0&-\lambda&1&0\\0&0&-\lambda&1\\0&1&0&-\lambda\end{vmatrix}-$

$\begin{vmatrix}1&0&0&0\\0&-\lambda&1&0\\0&0&-\lambda&1\\0&1&0&-\lambda\end{vmatrix}=\lambda^2\begin{vmatrix}-\lambda&1&0\\0&-\lambda&1\\1&0&-\lambda\end{vmatrix}-\begin{vmatrix}-\lambda&1&0\\0&-\lambda&1\\1&0&-\lambda\end{vmatrix}=\lambda^2(1-\lambda^3)-(1-\lambda^3)=-\lambda^5+\lambda^3+\lambda^2-1=0$

This can be factored by grouping, so $-\lambda^3(\lambda^2-1)+(\lambda^2-1)=-(\lambda^2-1)(\lambda^3-1)=0$ so the eigenvalues are $\lambda=\pm 1,\frac{-1\pm i\sqrt{3}}{2}$ with 1 as a repeated eigenvalue. Mathematica writes the roots this way:

$$\{\{x\to -1\},\{x\to 1\},\{x\to 1\},\{x\to -\sqrt[3]{-1}\},\{x\to(-1)^{2/3}\}\}$$

which seems a tad peculiar. If you ask for the Eigensystem[] it reports the expected eigenvalues:

$$-1 \quad 1 \quad 1 \quad \frac{1}{2}(-1+i\sqrt{3}) \quad \frac{1}{2}(-1-i\sqrt{3})$$

and corresponding eigenvectors:

$$\{-1,1,0,0,0\} \quad \{0,0,1,1,1\} \quad \{1,1,0,0,0\} \quad \left\{0,0,\frac{-1+i\sqrt{3}}{2},\frac{-1-i\sqrt{3}}{2},1\right\} \quad \left\{0,0,\frac{-1-i\sqrt{3}}{2},\frac{-1+i\sqrt{3}}{2},1\right\}$$

These shouldn't be too hard to verify. For $\lambda=-1, A+I=\begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&1&0&1\end{bmatrix}\sim\begin{bmatrix}1&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}$

So the null space is spanned by $(1,-1,0,0,0)$.

For $\lambda=1, A-I=\begin{bmatrix}-1&1&0&0&0\\1&-1&0&0&0\\0&0&-1&1&0\\0&0&0&-1&1\\0&0&1&0&-1\end{bmatrix}\sim\begin{bmatrix}1&-1&0&0&0\\0&0&1&0&-1\\0&0&0&1&-1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$ whose null space is spanned by $\left\{\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\1\\1\end{bmatrix}\right\}$

(note that rows 2 and 5, as there are no pivots in those columns, comprise I_2 while the negation of the elements of those non-pivot columns fill in the rest.)

As for the complex conjugate eigenvalues, well, for

$$\lambda=\frac{-1+i\sqrt{3}}{2}, A-\left(\frac{-1+i\sqrt{3}}{2}\right)I=\begin{bmatrix}\frac{1-i\sqrt{3}}{2} & 1 & 0 & 0 & 0\\ 1 & \frac{1-i\sqrt{3}}{2} & 0 & 0 & 0\\ 0 & 0 & \frac{1-i\sqrt{3}}{2} & 1 & 0\\ 0 & 0 & 0 & \frac{1-i\sqrt{3}}{2} & 1\\ 0 & 0 & 1 & 0 & \frac{1-i\sqrt{3}}{2}\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & \frac{1}{2}\left(1-i\sqrt{3}\right)\\ 0 & 0 & 0 & 1 & \frac{1}{4}\left(i+\sqrt{3}\right)^2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

whose null space is spanned by $\left(0,0,\frac{-1+i\sqrt{3}}{2},\frac{-1-i\sqrt{3}}{2},1\right)$ and for conjugate eigenvalue,

$$\lambda=\frac{-1-i\sqrt{3}}{2}, A-\left(\frac{-1-i\sqrt{3}}{2}\right)I=\begin{bmatrix}\frac{1+i\sqrt{3}}{2} & 1 & 0 & 0 & 0\\ 1 & \frac{1+i\sqrt{3}}{2} & 0 & 0 & 0\\ 0 & 0 & \frac{1+i\sqrt{3}}{2} & 1 & 0\\ 0 & 0 & 0 & \frac{1+i\sqrt{3}}{2} & 1\\ 0 & 0 & 1 & 0 & \frac{1+i\sqrt{3}}{2}\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & \frac{1}{2}(1+i\sqrt{3})\\ 0 & 0 & 0 & 1 & \frac{1}{4}(-i+\sqrt{3})^2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

whose null space is spanned by $\left(0,0,\frac{-1-i\sqrt{3}}{2},\frac{-1+i\sqrt{3}}{2},1\right)$. Thus the diagonalization is almost as shown below…apparently there are some problems with scaling (the upper left block has a different scale than the lower right block) and again the permutation thing…somehow the 3rd row remains to be pushed to the bottom…

$$\begin{bmatrix}0 & 2 & 0 & 0 & 0\\ 2 & 0 & 0 & 0 & 0\\ 0 & 0 & 3 & 0 & 0\\ 0 & 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 3 & 0\end{bmatrix}=\begin{bmatrix}-1 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 1\\ 1 & 1 & 0 & 0 & 0\\ 0 & 0 & \frac{-1+i\sqrt{3}}{2} & \frac{-1-i\sqrt{3}}{2} & 1\\ 0 & 0 & \frac{-1-i\sqrt{3}}{2} & \frac{-1+i\sqrt{3}}{2} & 1\end{bmatrix}^T\begin{bmatrix}-1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & \frac{-1+i\sqrt{3}}{2} & 0\\ 0 & 0 & 0 & 0 & \frac{-1-i\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}-1 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 1\\ 1 & 1 & 0 & 0 & 0\\ 0 & 0 & \frac{-1+i\sqrt{3}}{2} & \frac{-1-i\sqrt{3}}{2} & 1\\ 0 & 0 & \frac{-1-i\sqrt{3}}{2} & \frac{-1+i\sqrt{3}}{2} & 1\end{bmatrix}$$

(c) Express the eigenvalues in polar form.

SOLN: $\frac{-1\pm i\sqrt{3}}{2}=\cos\left(\pm\frac{2\pi}{3}\right)+i\sin\left(\pm\frac{2\pi}{3}\right)=\exp\left(\pm\frac{2\pi}{3}i\right)$

8. If $AS=S\begin{bmatrix}2 & 0 & 0\\ 0 & 3/2 & 0\\ 0 & 0 & 1/6\end{bmatrix}$ and S is invertible.
   a. What is the determinant of A?

      SOLN: $|A|=\left|S\begin{bmatrix}2 & 0 & 0\\ 0 & 3/2 & 0\\ 0 & 0 & 1/6\end{bmatrix}S^{-1}\right|=|S|\begin{vmatrix}2 & 0 & 0\\ 0 & 3/2 & 0\\ 0 & 0 & 1/6\end{vmatrix}|S^{-1}|=|S|\begin{vmatrix}2 & 0 & 0\\ 0 & 3/2 & 0\\ 0 & 0 & 1/6\end{vmatrix}\frac{1}{|S|}=\frac{1}{2}$
   b. What are the eigenvalues of $A-I$? Of $A-A^{-1}$? And of A^{-1}?

      SOLN: $A=S\Lambda S^{-1}\Leftrightarrow S^{-1}(A-I)S=\Lambda-I\Leftrightarrow A-I=S(\Lambda-I)S^{-1}$ so the eigenvalue matrix for $A-I$ is $\begin{bmatrix}1 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & -5/6\end{bmatrix}$ and the eigenvalues of $A-I$ are 1, ½ and –5/6.

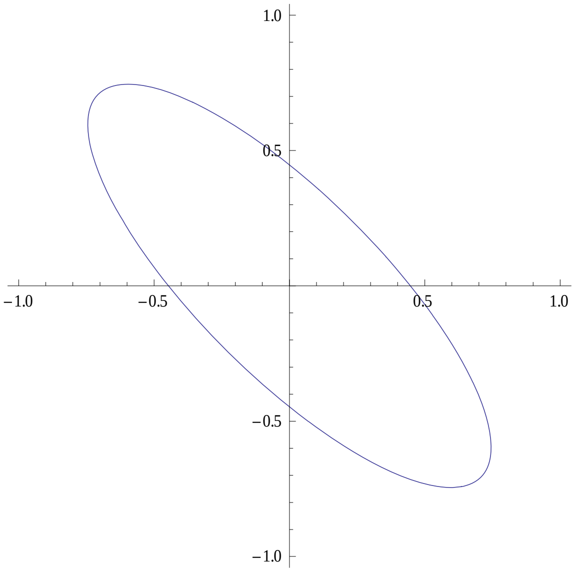
      Similarly, $A=S\Lambda S^{-1}\Leftrightarrow S^{-1}(A-A^{-1})S=\Lambda-\Lambda^{-1}\Leftrightarrow A-A^{-1}=S(\Lambda-\Lambda^{-1})S^{-1}$ so the eigenvalues of $A-A^{-1}$ are 3/2, 13/6 and 37/6. The eigenvalues of A^{-1} are ½, 2/3 and 6.
      a. Find a factored form of e^{tA} in terms of S and the diagonal matrix above.

$$e^{tA}=\sum_{n=0}^{\infty}\frac{A^n t^n}{n!}=S\sum_{n=0}^{\infty}\frac{\Lambda^n t^n}{n!}S^{-1}=S\begin{bmatrix}\sum_{n=0}^{\infty}\frac{(2t)^n}{n!} & 0 & 0\\ 0 & \sum_{n=0}^{\infty}\frac{(3t/2)^n}{n!} & 0\\ 0 & 0 & \sum_{n=0}^{\infty}\frac{(t/6)^n}{n!}\end{bmatrix}S^{-1}=S\begin{bmatrix}e^{2t} & 0 & 0\\ 0 & e^{\frac{3t}{2}} & 0\\ 0 & 0 & e^{\frac{t}{6}}\end{bmatrix}S^{-1}$$

b. Calculate the trace of A and then the trace of e^{tA}.
SOLN: The trace of A is the sum of its eigenvalues: 11/3. The trace of e^{tA} is then $e^{2t}+e^{3t/2}+e^{t/6}$

9. The quadratic $5x^2+8xy+5y^2=1$ describes a tilted ellipse in the *xy*-plane.
   a. Use eigenvalue methods to find the lengths of the major and minor axes.
      SOLN: $5x^2+8xy+5y^2=[x,y]\begin{bmatrix}5&4\\4&5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=1$. The characteristic polynomial $\lambda^2-10\lambda+9=(\lambda-9)(\lambda-1)$ has eigenvalue 9 with eigenvector (1,1) and eigenvalue 1 with eigenvector (–1, 1) so the diagonalization is $\begin{bmatrix}5&4\\4&5\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}9&0\\0&1\end{bmatrix}\begin{bmatrix}1&1\\-1&1\end{bmatrix}=\begin{bmatrix}\cos\pi/4&-\sin\pi/4\\ \sin\pi/4&\cos\pi/4\end{bmatrix}\begin{bmatrix}9&0\\0&1\end{bmatrix}\begin{bmatrix}\cos\pi/4&\sin\pi/4\\-\sin\pi/4&\cos\pi/4\end{bmatrix}$ Thus the ellipse is congruent to $9x^2+y^2=1$ whose major axis has length 1 and the minor axis has length 1/3.
   b. Find the directions of the axes of this tilted ellipse in the *xy*-plane
      SOLN: The first rotation is clockwise, so the major axis is pointed in the direction of

$$\text{ContourPlot}[5x^2+8xy+5y^2==1,\{x,-1,1\},\{y,-1,1\},\text{Axes}\to\text{True},\text{Frame}\to\text{False}]$$



10. One important use of SVD is in finding orthonormal bases for some or all of the four fundamental subspaces. Other uses include finding the rank of a matrix and solving least squares problems. You can extract subspace information by selecting appropriate columns from U or V. The code below shows Octave output for a specific “magic matrix.” Use the SVD output to find an orthonormal basis for the column space $C(A)$ and the nullspace $N(A)$. What is the rank of A?
    SOLN:

```
A=magic(4) % matrix in question
A =
16 2 3 13
5 11 10 8
9 7 6 12
4 14 15 1
octave:5> [u, s, v] = svd(A)

u =
-0.50000 0.67082 0.50000 -0.22361
-0.50000 -0.22361 -0.50000 -0.67082
-0.50000 0.22361 -0.50000 0.67082
-0.50000 -0.67082 0.50000 0.22361
s =
Diagonal Matrix
3.4000e+001 0 0 0
0 1.7889e+001 0 0
0 0 4.4721e+000 0
0 0 0 8.3456e-016
```

```
v =
-0.50000 0.50000 0.67082 -0.22361
-0.50000 -0.50000 -0.22361 -0.67082
-0.50000 -0.50000 0.22361 0.67082
-0.50000 0.50000 -0.67082 0.22361
```

Clearly that last eigenvalue is machine zero. Well, let's be as honest as we can. The column vectors of U are the eigenvectors of $AA^T = \begin{bmatrix} 438 & 236 & 332 & 150 \\ 236 & 310 & 278 & 332 \\ 332 & 278 & 310 & 236 \\ 150 & 332 & 236 & 438 \end{bmatrix}$. The eigenvalues of AA^T are values of λ that make $AA^T - \lambda I$ singular. That is, so that $|AA^T - \lambda I\ | = \begin{vmatrix} 438-\lambda & 236 & 332 & 150 \\ 236 & 310-\lambda & 278 & 332 \\ 332 & 278 & 310-\lambda & 236 \\ 150 & 332 & 236 & 438-\lambda \end{vmatrix} =$
$\lambda^4 - 1496\lambda^3 + 399440\lambda^2 - 7398400\lambda = \lambda(\lambda - 34)(\lambda - 320)(\lambda - 1156)$
The eigenvector for $\lambda = 1156$ is the unit vector $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$ that spans the null space of $AA^T - 1156I =$
$\begin{bmatrix} -718 & 236 & 332 & 150 \\ 236 & -846 & 278 & 332 \\ 332 & 278 & -846 & 236 \\ 150 & 332 & 236 & -718 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
The eigenvector for $\lambda = 320$ is the unit vector $\left[\frac{3\sqrt{5}}{10}, \frac{-\sqrt{5}}{10}, \frac{\sqrt{5}}{10}, \frac{-3\sqrt{5}}{10}\right]$ that spans the null space of $AA^T -$
$320I = \begin{bmatrix} 118 & 236 & 332 & 150 \\ 236 & -10 & 278 & 332 \\ 332 & 278 & -10 & 236 \\ 150 & 332 & 236 & 118 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The eigenvector for $\lambda = 34$ is the unit vector $\left[\frac{3\sqrt{5}}{10}, \frac{-\sqrt{5}}{10}, \frac{\sqrt{5}}{10}, \frac{-3\sqrt{5}}{10}\right]$ that spans the null space of $AA^T - 34I =$
$\begin{bmatrix} 404 & 236 & 332 & 150 \\ 236 & 276 & 278 & 332 \\ 332 & 278 & 276 & 236 \\ 150 & 332 & 236 & 404 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ so the nullspace is empty. Note that $34^2 = 1156$. Hmmm.

Since $AA^T \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ we might instead use $\left[-\frac{\sqrt{5}}{10}, -\frac{3\sqrt{5}}{10}, \frac{3\sqrt{5}}{10}, \frac{\sqrt{5}}{10}\right]$ – but that's for the zero

eigenvector. I do not fully grasp what's going on here…I suspect it's something deep.
In any case, let's turn our attention to V. The column vectors are the eigenvectors of

$A^TA = \begin{bmatrix} 378 & 206 & 212 & 360 \\ 206 & 370 & 368 & 212 \\ 212 & 368 & 370 & 206 \\ 360 & 212 & 206 & 378 \end{bmatrix}$. Since A is symmetric, $A^TA = AA^T$ and the singular value decomposition has well…

$$A=\begin{bmatrix}\frac{1}{2} & \frac{3\sqrt{5}}{10} & -\frac{1}{2} & -\frac{\sqrt{5}}{10}\\ \frac{1}{2} & -\frac{\sqrt{5}}{10} & \frac{1}{2} & -\frac{3\sqrt{5}}{10}\\ \frac{1}{2} & \frac{\sqrt{5}}{10} & \frac{1}{2} & \frac{3\sqrt{5}}{10}\\ \frac{1}{2} & -\frac{3}{2\sqrt{5}} & -\frac{1}{2} & \frac{\sqrt{5}}{10}\end{bmatrix}\begin{bmatrix}34 & 0 & 0 & 0\\ 0 & 8\sqrt{5} & 0 & 0\\ 0 & 0 & 2\sqrt{5} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & -\frac{3\sqrt{5}}{10} & -\frac{\sqrt{5}}{10}\\ \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{5}}{10} & -\frac{3\sqrt{5}}{10}\\ \frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{5}}{10} & \frac{3\sqrt{5}}{10}\\ \frac{1}{2} & \frac{1}{2} & \frac{3\sqrt{5}}{10} & \frac{\sqrt{5}}{10}\end{bmatrix}$$

Where $U=\begin{bmatrix}\frac{1}{2} & \frac{3}{2\sqrt{5}} & -\frac{1}{2} & -\frac{1}{2\sqrt{5}}\\ \frac{1}{2} & -\frac{1}{2\sqrt{5}} & \frac{1}{2} & -\frac{3}{2\sqrt{5}}\\ \frac{1}{2} & \frac{1}{2\sqrt{5}} & \frac{1}{2} & \frac{3}{2\sqrt{5}}\\ \frac{1}{2} & -\frac{3}{2\sqrt{5}} & -\frac{1}{2} & \frac{1}{2\sqrt{5}}\end{bmatrix}\approx\begin{bmatrix}0.5 & 0.67082 & -0.5 & -0.223607\\ 0.5 & -0.223607 & 0.5 & -0.67082\\ 0.5 & 0.223607 & 0.5 & 0.67082\\ 0.5 & -0.67082 & -0.5 & 0.223607\end{bmatrix}$ is pretty much what Octave got.

$V^T=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & -\frac{3}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}}\\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2\sqrt{5}} & -\frac{3}{2\sqrt{5}}\\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}}\\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}}\end{bmatrix}\approx\begin{bmatrix}0.5 & 0.5 & -0.67082 & -0.223607\\ 0.5 & -0.5 & 0.223607 & -0.67082\\ 0.5 & -0.5 & -0.223607 & 0.67082\\ 0.5 & 0.5 & 0.67082 & 0.223607\end{bmatrix}$ is likewise.

As to S , $\sigma_1=\sqrt{\lambda_1}=\sqrt{1156}=34$, $\sigma_2=\sqrt{\lambda_2}=\sqrt{320}=16\sqrt{5}\approx 17.889$ but $\sigma_3=\sqrt{\lambda_3}=\sqrt{34}\approx 5.831$ doesn't jibe with what the machine algorithms are getting, which is $\sigma_3=2\sqrt{5}$ Go figure. While you're at it, go transfer to university and find yourselves some answer to this!

11. Suppose that A is 5 by 3 and $A=U\Sigma V^T$ is its SVD, where

$$\Sigma=\begin{bmatrix}4 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

a. Find the eigenvalues of A^TA using $A=U\Sigma V^T$.

SOLN: $\Sigma^T\Sigma=\begin{bmatrix}16 & 0 & 0 & 0 & 0\\ 0 & 4 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$ suggests that the eigenvalues are 16, 4, 1

b. Remember properties of orthogonal matrices and evaluate the determinants, $|AA^T|$ and $|A^TA|$.

SOLN: Clearly U will be singular, while V will not, so $|AA^T|=0$ while $|A^TA|=64$.

12. Write $(x_1+x_2+x_3+x_4)^2+(x_1-x_2-x_3)^2=x^TAx$ in terms of a symmetric matrix A and vector $x^T=[x_1,x_2,x_3,x_4]$. What is the rank of A?

SOLN: Take $A=B^TB=\begin{bmatrix}1 & 1\\ 1 & -1\\ 1 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & -1 & 0\end{bmatrix}=[x_1,x_2,x_3,x_4]\begin{bmatrix}1 & 1\\ 1 & -1\\ 1 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=$

$[(x_1+x_2+x_3+x_4)\quad(x_1-x_2-x_3)]\begin{bmatrix}(x_1+x_2+x_3+x_4)\\ (x_1-x_2-x_3)\end{bmatrix}=(x_1+x_2+x_3+x_4)^2+(x_1-x_2-x_3)^2$

Clearly the rank of A is 2.