

Give complete solutions to all problems. You can use technology to check your results, but be sure to detail how they are derived. Do not copy other people's work.

1. Find the LU factorization of $A = \begin{bmatrix} 2 & 9 & 8 & 7 \\ 10 & 46.5 & 44 & 38 \\ 6 & 28.5 & \frac{88}{3} & 26 \\ 0 & 0 & \frac{20}{3} & 11.25 \end{bmatrix}$ and use it to compute the determinant, $|A^n|$ in terms of n .

2. Assuming we do not perform any row permutations, what is the 5th and last pivot when transforming the following matrix A into an upper triangular matrix by row eliminations? Explain.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 2 & 4 & 6 & 0 \\ -1 & -2 & 2 & 4 & 0 \\ 5 & 4 & 3 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 \end{bmatrix}$$

3. Is there an LDL^T factorization of $A = \begin{bmatrix} 16 & 2 & 9 & 13 \\ 2 & 11 & 5 & 13 \\ 9 & 9 & 6 & 5 \\ 13 & 14 & 5 & 1 \end{bmatrix}$. Explain.

4. Find the Cholesky factorization of $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. Is A positive definite? Explain.

5. The implicit diagonalization $AS = S\Lambda$ occurs in eigenvalue problems, where Λ is the diagonal matrix of eigenvalues.
- Describe the significance of the individual columns of S in relation to the diagonal entries in Λ .
 - Describe two uses of the diagonalization factorization.

6. Let $A = \begin{pmatrix} -3 & 2 & 4 \\ 2 & -6 & 2 \\ 4 & 2 & -3 \end{pmatrix}$
- Find the eigenvalues of A .
 - Find an orthogonal matrix P such that $P^T A P$ is diagonal.
 - Find an expression for $\exp(A)$.

7. (a) Argue that the eigenvalues of a permutation matrix must all have a modulus equal to 1.

(b) What are the eigenvalues of $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$?

- (c) Express the eigenvalues in polar form.

8. If $AS = S \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 1/6 \end{bmatrix}$ and S is invertible.
- What is the determinant of A ?
 - What are the eigenvalues of $A - I$? Of $A - A^{-1}$? And of A^{-1} ?
 - Find a factored form of e^{tA} in terms of S and the diagonal matrix above.
 - Calculate the trace of A and then the trace of e^{tA} .
9. The quadratic $5x^2 + 8xy + 5y^2 = 1$ describes a tilted ellipse in the xy -plane.
- Use eigenvalue methods to find the lengths of the major and minor axes.
 - Find the directions of the axes of this tilted ellipse in the xy -plane
10. One important use of SVD is in finding orthonormal bases for some or all of the four fundamental subspaces. Other uses include finding the rank of a matrix and solving least squares problems. You can extract subspace information by selecting appropriate columns from U or V . The code below shows Octave output for a specific “magic matrix.” Use the SVD output to find an orthonormal basis for the column space $\mathcal{C}(A)$ and the nullspace $\mathcal{N}(A)$. What is the rank of A ?

```
A=magic(4) % matrix in question
A =
16 2 3 13
5 11 10 8
9 7 6 12
4 14 15 1
octave:> help svd
-- Loadable Function: S = svd (A)
-- Loadable Function: [U, S, V] = svd (A)
-- Loadable Function: [U, S, V] = svd (A, ECON)
Compute the singular value decomposition of A
A = U*S*V'
The function 'svd' normally returns only the vector of singular
values. When called with three return values, it computes U, S,
and V. For example,
svd (hilb (3))
returns
ans =
1.4083189
0.1223271
0.0026873
and
[u, s, v] = svd (hilb (3))
returns
u =
-0.82704 0.54745 0.12766
-0.45986 -0.52829 -0.71375
-0.32330 -0.64901 0.68867
s =
1.40832 0.00000 0.00000
0.00000 0.12233 0.00000
0.00000 0.00000 0.00269
v =
-0.82704 0.54745 0.12766
-0.45986 -0.52829 -0.71375
-0.32330 -0.64901 0.68867
If given a second argument, `svd' returns an economy-sized
decomposition, eliminating the unnecessary rows or columns of U or V.
```

11. Suppose that A is 5 by 3 and $A = U\Sigma V^T$ is its SVD, where

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Find the eigenvalues of $A^T A$ using $A = U\Sigma V^T$.
- Remember properties of orthogonal matrices and evaluate the determinants, $|AA^T|$ and $|A^T A|$.

12. Write $(x_1 + x_2 + x_3 + x_4)^2 + (x_1 - x_2 - x_3)^2 = x^T A x$ in terms of a symmetric matrix A and vector $x^T = [x_1, x_2, x_3, x_4]$. What is the rank of A ?