Math 2B – Linear Algebra – Test 2 – S13 Name______________________________

Write all responses on separate paper. Show your work for credit.

1. Construct a matrix whose
   a. null space consists of all combinations of (1,3,3,1) and (1,2,1,0).
   b. Left null space consists of all combinations of (1,2,1,0) and (1,1,0,0).
2. Construct a 4by4 matrix whose column space is the same as its null space. Or show that it's not possible.
3. Show that
$$\begin{bmatrix}1 & 3 & 3 & 1\\ 1 & 4 & 6 & 4\\ 1 & 5 & 10 & 10\end{bmatrix}\vec{x}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$
   Has infinitely many solutions and find them all.
4. Are there values for *u* and *v* so that
$$\begin{bmatrix}1 & 1 & 1 & 0 & 2\\ 0 & u & 4 & 8 & 16\\ 0 & 0 & 0 & v & 16\end{bmatrix}$$
   Has rank 2?
5. The cosine space $\mathbb{F}_4$ contains all combinations of
$$y(x)=Acos(x)+Bcos(2x)+Ccos(3x)+Dcos(4x)$$
   Find a basis for the subspace with $y(0)=0$.
6. Discuss the four subspaces of $\mathbb{R}^4$ associated with
$$A=\begin{bmatrix}1 & 2 & 1 & 0\\ 1 & 3 & 3 & 1\\ 3 & 7 & 6 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} \text{ and } I_4+A$$
7. If V is the space of vectors in $\mathbb{R}^4$ satisfying
   $x_1+3x_2+3x_3+x_4=0$ and $x_1+4x_2+6x_3+4x_4=0$
   Write a basis for its orthogonal complement $V^{\perp}$.
8. $p = A\hat{x}$ is the vector in C(A) nearest to a given vector *b*. If A has independent columns, what equation determines $\hat{x}$? What are all the vectors perpendicular to the error $= b - A\hat{x}$? What goes wrong if the columns of A are dependent?
9. Use matrix methods to find the fourth degree polynomial which is the best fit for the data: (0,1), (1,3), (2,2), (3,5), (4,4),(5,7).
10. Use the Gram Schmidt process to find an orthonormal basis for the column space of
$$\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 1 & 2 & 1 & 0 & 0\\ 1 & 3 & 3 & 1 & 0\\ 1 & 4 & 6 & 4 & 1\end{bmatrix}$$
11. Let V be a finite-dimensional, nonzero vector space over a field $\mathbb{F}$($\mathbb{F}$ could be either $\mathbb{R}$ or $\mathbb{C}$). An inner product on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u,v\rangle$ and has the following properties:
   - **Positivity.**
     $\langle v,v\rangle \geq 0$ for all $v \in V$;
   - **Definiteness.**
     $\langle v,v\rangle = 0$ if and only if $v=0$;
   - **Additivity**
     $\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle$ for all $u,v,w \in V$;

- **Homogeneity**

  $\langle au, v\rangle = a\langle u, v\rangle$ for all $a \in F$ and all $u, v \in V$;
- **Conjugate Symmetry**

  $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$;

Recall that every real number is its complex conjugate, so if $\mathbb{F} = \mathbb{R}$, then the Conjugate Symmetry condition is automatic.

Let $C[0,1]$ denote the real-valued continuous functions on [0,1]. Define an inner product by $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)dx$

a. Show that this is an inner product space.

b. Apply the Gram-Schmidt method to get an orthonormal basis for subspace spanned by $\{1, \sin(x), \cos(x)\}$

SOLNS:

1. Construct a matrix whose
   a. null space consists of all combinations of

      ANS: We want to find a matrix N such that $N\begin{bmatrix}1 & 1\\3 & 2\\3 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

      Of course you could also look at it this way:
      $$\begin{bmatrix}1 & 3 & 3 & 1\\1 & 2 & 1 & 0\end{bmatrix}N^T = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$
      $$\begin{bmatrix}1 & 3 & 3 & 1\\1 & 2 & 1 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -3 & -2\\0 & 1 & 2 & 1\end{bmatrix}$$
      shows that $N = \begin{bmatrix}3 & -2 & 1 & 0\\2 & -1 & 0 & 1\end{bmatrix}$ is a matrix with rank 2 and the null space has dimension $n - r = 4 - 2 = 2$ including the two independent vectors, (1,3,3,1) and (1,2,1,0).

   b. Left null space consists of all combinations of (1,2,1,0) and (1,1,0,0).

      SOLN: Basically the same deal. We want to find a matrix N such that $N\begin{bmatrix}1 & 1\\2 & 1\\1 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

      Of course you could also look at it this way:
      $$\begin{bmatrix}1 & 2 & 1 & 0\\1 & 1 & 0 & 0\end{bmatrix}N^T = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

   c.
      $$\begin{bmatrix}1 & 2 & 1 & 0\\1 & 1 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1 & 0\\0 & 1 & 1 & 0\end{bmatrix}$$
      shows that $N = \begin{bmatrix}1 & -1 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$ is a matrix with rank 2 and the null space has dimension $n - r = 4 - 2 = 2$ including the two independent vectors, (1,2,1,0) and (1,1,0,0).

      Thus $A = \begin{bmatrix}1 & 0\\-1 & 0\\1 & 0\\0 & 1\end{bmatrix}$ is a matrix whose left null space contains the specified vectors.

2. Construct a 4by4 matrix whose column space is the same as its null space. Or show that it's not possible.
SOLN: If $n = 4$ and the dimensions of $C(A)$ are the same as the dimensions of $N(A)$ then they are both 2 dimensional spaces embedded in 4 dimensional spaces. The column space is a subspace of the output and the null space is a subspace of the input. So they are both essentially isomorphic to $\mathbb{R}^2$ (as any 2-dimensional space is) but they are also both embedded in $\mathbb{R}^4$ – so the question is , can these be the same space embedded in $\mathbb{R}^4$? I had to resort to (enlightened) guess and check and I finally struck upon $= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The column space is spanned by the basis $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ and these are also two independent vectors in the null space. So that does it!

3. Show that

$$\begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \\ 1 & 5 & 10 & 10 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Has infinitely many solutions and find them all.
SOLN: Start by getting reduced row-echelon form of the augmented matrix:

$$\begin{bmatrix} 1 & 3 & 3 & 1 & 1 \\ 1 & 4 & 6 & 4 & 2 \\ 1 & 5 & 10 & 10 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & 1 & 1 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 2 & 7 & 9 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -6 & -8 & -2 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 10 & -2 \\ 0 & 1 & 0 & -6 & 1 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix}$$

Now the nullspace matrix is $N = s = \begin{bmatrix} -10 \\ 6 \\ -3 \\ 1 \end{bmatrix}$ and this also the "special" solution (according to Strang.) Now a particular solution is given by $x_p = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and thus the general solution is

$$x = x_p + x_n = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -10 \\ 6 \\ -3 \\ 1 \end{bmatrix}$$

4. Are there values for *u* and *v* so that

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & u & 4 & 8 & 16 \\ 0 & 0 & 0 & v & 16 \end{bmatrix}$$

Has rank 2?
SOLN: Assume $u \neq 0$ and do a row operation to move the matrix toward reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 1 - 4/u & -8/u & 2 - 16/u \\ 0 & 1 & 4/u & 8/u & 16/u \\ 0 & 0 & 0 & v & 16 \end{bmatrix}$$

Clearly, no matter what *v* is, this will have rank 3. If *u* = 0, that's still true. Same thing for *v*. So basically, we want to reduce the number of pivots to 2, but there's no way to do that. So, no, there are no values for *u* and *v* so that the matrix has rank 2.

5. The cosine space $\mathbb{F}_4$ contains all combinations of

$$y(x) = Acos(x) + Bcos(2x) + Ccos(3x) + Dcos(4x)$$

Find a basis for the subspace with $y(0)=0$.
SOLN: First let's be sure that $\mathbb{F}_4$ is, in fact, a vector space. Clearly it's closed under vector addition and scalar multiplication. All the other axioms are satisfied essentially by inheritance from $\mathbb{R}$.
Now let's see that the subset with $y(0)=0$ is also a vector space. Clearly, again this space is closed under vector addition and scalar multiplication…so that pretty much does it.
Now if $\cos(4x) =$ If $y(0)=A+B+C+D=0$, which means there are only 3 degrees of freedom, knowing the values of A, B, and C determines the value of $D=-(A+B+C)$. So we could write

$$\begin{aligned}y(x) &= A\cos x + B\cos 2x + C\cos 3x - (A+B+C)\cos 4x \\ &= A(\cos x - \cos 4x) + B(\cos 2x - \cos 4x) + C(\cos 3x - \cos 4x)\end{aligned}$$

Interestingly, you can use the sum to product identities to write this as
$y(x) = A\sin\frac{3x}{2}\sin\frac{5x}{2} + B\sin x\sin 3x + C\sin\frac{x}{2}\sin\frac{7x}{2}$ So you could take the basis as
$\{\cos x - \cos 4x, \cos 2x - \cos 4x, \cos 3x - \cos 4x\}$ or $\left\{\sin\frac{3x}{2}\sin\frac{5x}{2}, \sin x\sin 3x, \sin\frac{x}{2}\sin\frac{7x}{2}\right\}$.

6. Discuss the four subspaces of $\mathbb{R}^4$ associated with

$$A = \begin{bmatrix}1&2&1&0\\1&3&3&1\\3&7&6&1\\0&0&0&0\end{bmatrix} \quad \text{and} \quad I_4 + A$$

SOLN: $\begin{bmatrix}1&2&1&0\\1&3&3&1\\3&7&6&1\\0&0&0&0\end{bmatrix} \sim \begin{bmatrix}1&2&1&0\\0&1&2&1\\0&1&3&1\\0&0&0&0\end{bmatrix} \sim \begin{bmatrix}1&0&-3&-2\\0&1&2&1\\0&0&1&0\\0&0&0&0\end{bmatrix} \sim \begin{bmatrix}1&0&0&-2\\0&1&0&1\\0&0&1&0\\0&0&0&0\end{bmatrix}$ so the rank is 3 and the null space has dimension 1.

The column space has basis $\left\{\begin{bmatrix}1\\1\\3\\0\end{bmatrix}, \begin{bmatrix}2\\3\\7\\0\end{bmatrix}, \begin{bmatrix}1\\3\\6\\0\end{bmatrix}\right\}$, comprised of the columns of A that correspond to the pivot elements of the rref form of A.

The null space is spanned by the vector $\begin{bmatrix}2\\-1\\0\\1\end{bmatrix}$.

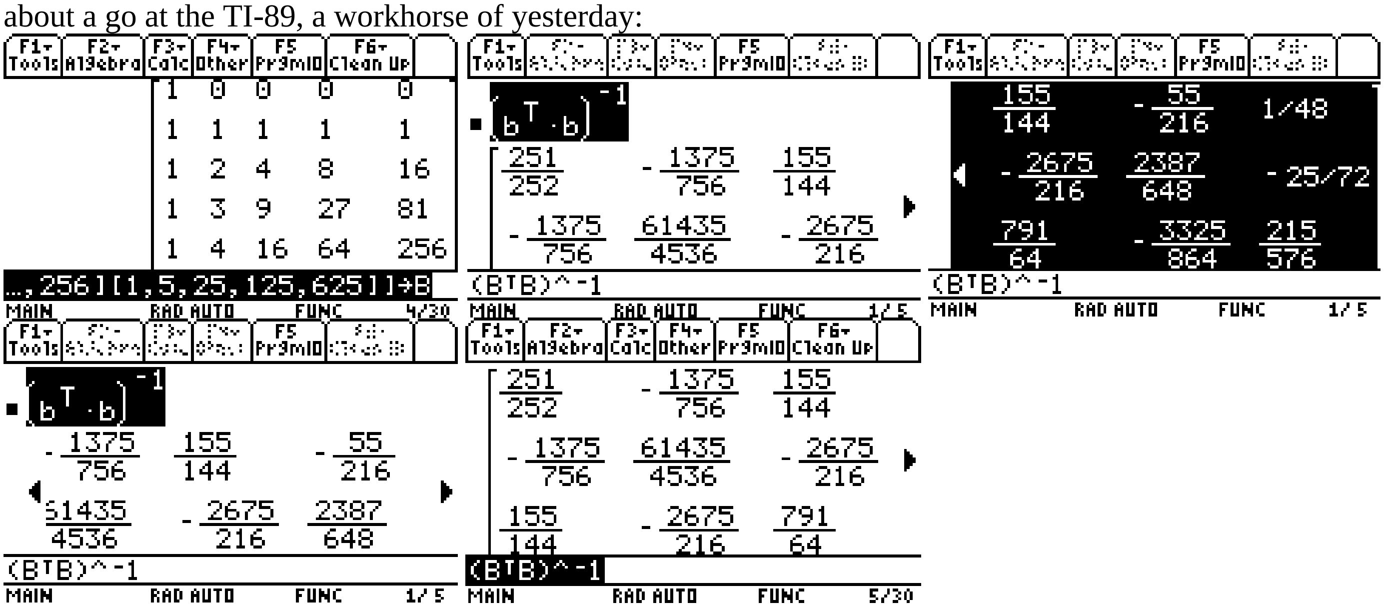
The row space is also 3-dimensional spanned by any of the first three vectors comprising the rows of a matrix row-equivalent to A (provided you haven't done some weird permutation.)
The left null space…hmmm, clearly it's one-dimensional. What vector spans the space? For that we need to look at $A^T = \begin{bmatrix}1&1&3&0\\2&3&7&0\\1&3&6&1\\0&1&1&0\end{bmatrix} \sim \begin{bmatrix}1&1&3&0\\0&1&1&0\\0&2&3&1\\0&1&1&0\end{bmatrix} \sim \begin{bmatrix}1&0&2&0\\0&1&1&0\\0&0&1&1\\0&0&0&0\end{bmatrix} \sim \begin{bmatrix}1&0&0&-2\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{bmatrix}$ so the left null space is spanned by (2,1,-1,1).
As to

$$I_4 + A = \begin{bmatrix}2&2&1&0\\1&4&3&1\\3&7&7&1\\0&0&0&1\end{bmatrix} \sim \begin{bmatrix}1&2&1&0\\0&0&1&1\\0&4&11/2&1\\0&0&0&1\end{bmatrix} \sim \begin{bmatrix}1&0&-7/4&-1/2\\0&0&1&1\\0&4&11/2&0\\0&0&0&1\end{bmatrix} \sim \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

So this matrix has full rank, meaning n = m = r = 4 and the null spaces consist of just the zero vector.

7. If V is the space of vectors in $\mathbb{R}^4$ satisfying
   $x_1 + 3x_2 + 3x_3 + x_4 = 0$ and $x_1 + 4x_2 + 6x_3 + 4x_4 = 0$
   Write a basis for its orthogonal complement $V^{\perp}$.
   SOLN: The trick here may be to rephrase the question in terms of our matrix language.
$$\begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -6 & -8 \\ 0 & 1 & 3 & 3 \end{bmatrix}$$
   V is actually the null space of this matrix, so it's orthogonal complement is the row space, whose basis can be taken as either $\left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \\ 4 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 1 \\ 0 \\ -6 \\ -8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 3 \end{bmatrix} \right\}$.
8. $p = A\hat{x}$ is the vector in C(A) nearest to a given vector b. If A has independent columns, what equation determines $\hat{x}$? What are all the vectors perpendicular to the error $= b - A\hat{x}$?
   What goes wrong if the columns of A are dependent?
   SOLN: $b - A\hat{x}$ is the error in approximating $b \approx A\hat{x}$ and is minimized when it orthogonal to the column space: $A^T(b - A\hat{x}) = 0 \Leftrightarrow \hat{x} = (A^T A)^{-1} A^T b$ and we can be sure that the inverse exists because the columns of A are independent.
9. Use matrix method to find the fourth degree polynomial which is the best fit for the data:
   (0,1), (1,3), (2,2), (3,5), (4,4),(5,7).
   SOLN: Obviously we want to use the least square projection method described in the text. The general form of the equation is $p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ and the system of equations corresponding to the given data is described by the matrix equation
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \\ 1 & 5 & 25 & 125 & 625 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \\ 4 \\ 7 \end{bmatrix}$$

The system involves 6 equations in 5 unknowns, so it's overdetermined and may not have a solution. We could do row operations by hand or work the formula from #8 by hand, but it's a bit of a bear, so let's turn to Octave. But first, since Octave doesn't (or can't figure out how to) produce factions in exact form, how about a go at the TI-89, a workhorse of yesterday:

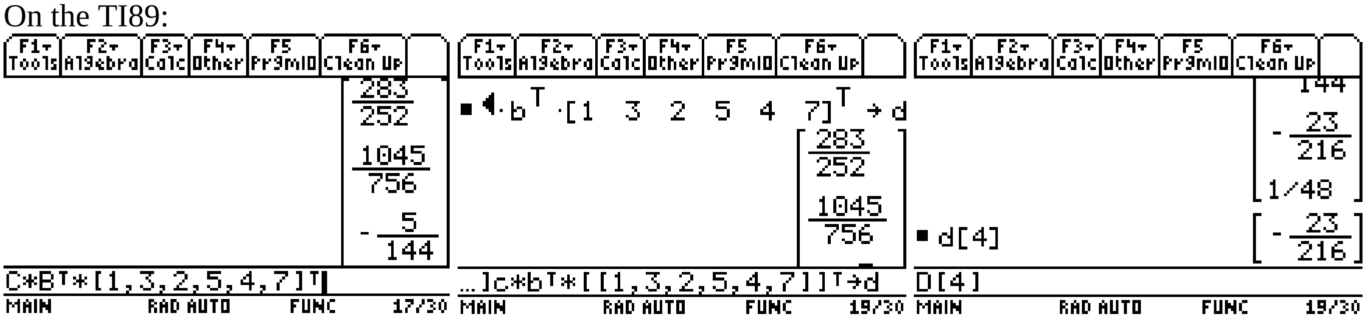


As these screenshots suggest, $(A^TA)^{-1} = \begin{bmatrix} \frac{251}{252} & -\frac{1375}{756} & \frac{155}{144} & -\frac{55}{216} & \frac{1}{48} \\ -\frac{1375}{756} & \frac{61435}{4536} & -\frac{2675}{216} & \frac{2387}{648} & -\frac{25}{72} \\ \frac{155}{144} & -\frac{2675}{216} & \frac{791}{64} & -\frac{3325}{864} & \frac{215}{576} \\ -\frac{55}{216} & \frac{2387}{648} & -\frac{3325}{864} & \frac{1595}{1296} & -\frac{35}{288} \\ \frac{1}{48} & -\frac{25}{72} & \frac{215}{576} & -\frac{35}{288} & \frac{7}{576} \end{bmatrix}$. How much fun is that? In Octave we have

```
octave:5> B = [1 0 0 0 0;1 1 1 1 1; 1 2 4 8 16; 1 3 9 27 81; 1 4 16 64
256; 1 5 25 125 625]
B =
1 0 0 0 0
1 1 1 1 1
1 2 4 8 16
1 3 9 27 81
1 4 16 64 256
1 5 25 125 625
octave:6> inverse(B'*B)
ans =
 0.996032 -1.818783   1.076389  -0.254630  0.020833
-1.818783  13.543871 -12.384259  3.683642 -0.347222
 1.076389 -12.384259  12.359375 -3.848380  0.373264
-0.254630  3.683642  -3.848380   1.230710 -0.121528
 0.020833  -0.347222   0.373264   -0.121528   0.012153
```

Ok now for $\hat{x} = (A^TA)^{-1}A^Tb =$3

On the TI89:



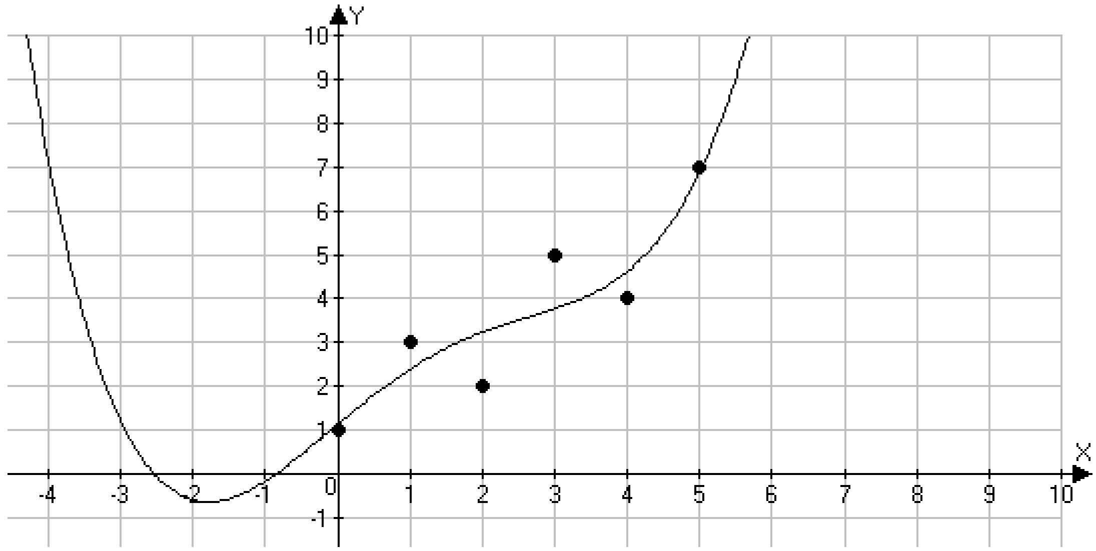
This shows that the least square 4th degree polynomial fit is $p(x) = \frac{1}{48}x^4 - \frac{23}{216}x^3 - \frac{5}{144}x^2 + \frac{1045}{756}x + \frac{283}{252}$

In Octave,

```
octave:7> xhat=inverse(B'*B)*B'*[1 3 2 5 4 7]'
xhat =
1.123016
1.382275
-0.034722
-0.106481
0.020833
```

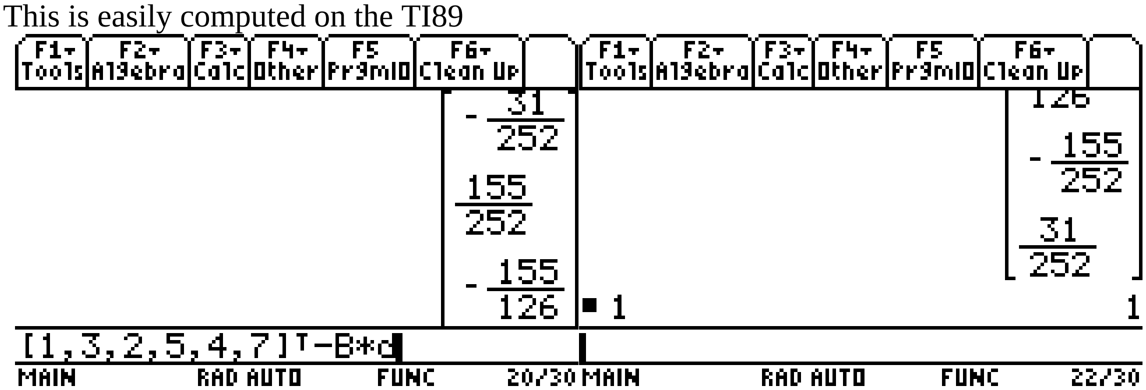
This is consistent with the TI89 since

$\frac{1}{48} \approx 0.020833, -\frac{23}{216} \approx -0.106481, -\frac{5}{144} \approx -0.034722, \frac{1045}{756} \approx 1.382275$ and $\frac{283}{252} \approx 1.123016$



Not a bad fit. The error is $b - A\hat{x} = \begin{bmatrix}1\\3\\2\\5\\4\\7\end{bmatrix} - \begin{bmatrix}1 & 0 & 0 & 0 & 0\\1 & 1 & 1 & 1 & 1\\1 & 2 & 4 & 8 & 16\\1 & 3 & 9 & 27 & 81\\1 & 4 & 16 & 64 & 256\\1 & 5 & 25 & 125 & 625\end{bmatrix}\begin{bmatrix}\frac{283}{252}\\\frac{1045}{756}\\-\frac{5}{144}\\-\frac{23}{216}\\\frac{1}{48}\end{bmatrix} = \begin{bmatrix}\frac{-31}{252}\\\frac{155}{252}\\-\frac{155}{126}\\-\frac{155}{252}\\\frac{31}{252}\end{bmatrix}$

This is easily computed on the TI89



or using Octave,

```
octave:9> [1 3 2 5 4 7]' - B*xhat
ans =
-0.12302
 0.61508
-1.23016
 1.23016
-0.61508
 0.12302
```

10. Use the Gram-Scmidt method to find an orthonormal basis for the column space of

$$\begin{bmatrix}1 & 0 & 0 & 0 & 0\\1 & 1 & 0 & 0 & 0\\1 & 2 & 1 & 0 & 0\\1 & 3 & 3 & 1 & 0\\1 & 4 & 6 & 4 & 1\end{bmatrix}$$

SOLN: Start with $A = a = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$, Then form *B* by subtracting from *b*, its projection onto *A*:

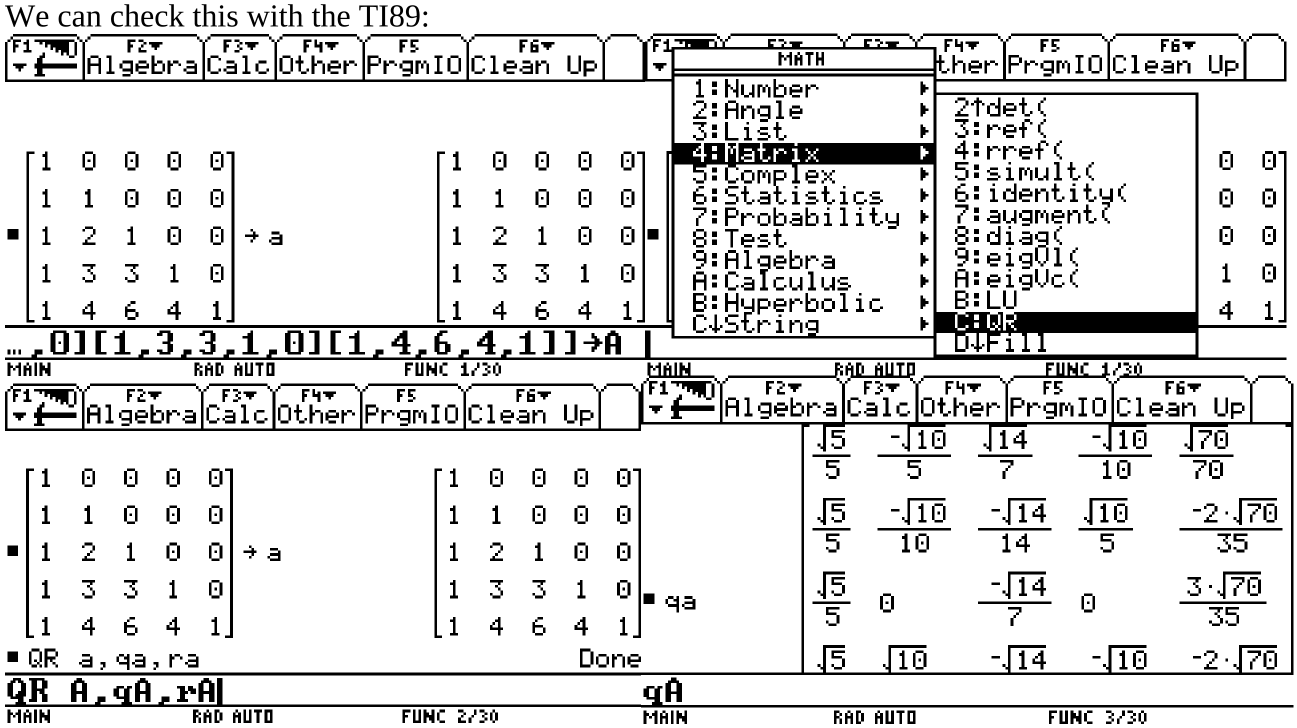
$B = b - \frac{A^Tb}{A^TA}A = \begin{bmatrix}0\\1\\2\\3\\4\end{bmatrix} - \frac{10}{5}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix}$, but don't stop there, now subtract from the third column vector its projection onto A and its projection onto B:

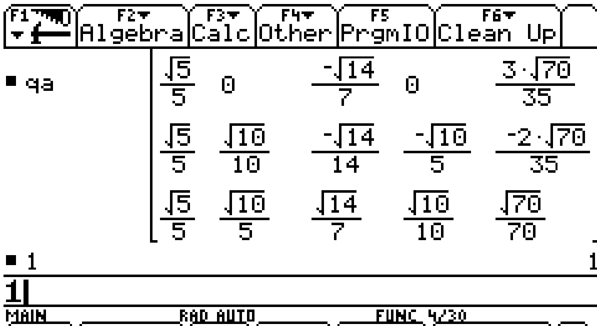
$C = c - \frac{A^Tc}{A^TA}A - \frac{B^Tc}{B^TB}B = \begin{bmatrix}0\\0\\1\\3\\6\end{bmatrix} - \frac{10}{5}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} - \frac{15}{10}\begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix} = \begin{bmatrix}1\\-\frac{1}{2}\\-1\\-\frac{1}{2}\\1\end{bmatrix}$, and continue with *D* in the same fashion:

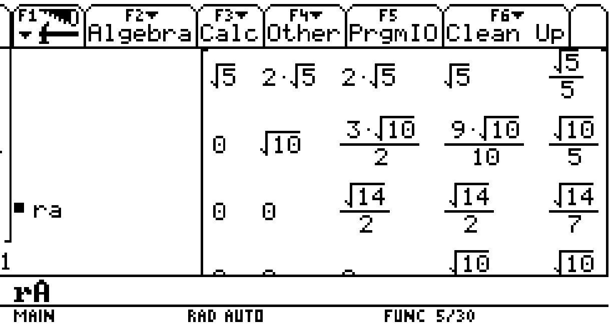
$D = d - \frac{A^Td}{A^TA}A - \frac{B^Td}{B^TB}B - \frac{C^Td}{C^TC}C = \begin{bmatrix}0\\0\\0\\1\\4\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} - \frac{9}{10}\begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix} - \begin{bmatrix}1\\-\frac{1}{2}\\-1\\-\frac{1}{2}\\1\end{bmatrix} = \begin{bmatrix}-1/5\\2/5\\0\\-2/5\\1/5\end{bmatrix}$. Finally,

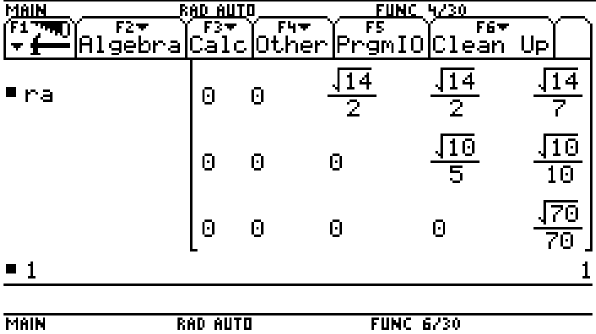
$$E = e - \frac{A^Te}{A^TA}A - \frac{B^Te}{B^TB}B - \frac{C^Te}{C^TC}C - \frac{D^Te}{D^TD}D = \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} - \frac{1}{5}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} - \frac{1}{5}\begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix} - \frac{2}{7}\begin{bmatrix}1\\-\frac{1}{2}\\-1\\-\frac{1}{2}\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-\frac{1}{5}\\\frac{2}{5}\\0\\-\frac{2}{5}\\\frac{1}{5}\end{bmatrix} = \begin{bmatrix}1/70\\-2/35\\3/35\\-2/35\\1/70\end{bmatrix}.$$

We can check this with the TI89:









Evidently, the *QR* factorization of *A* is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}}{5} & -\frac{\sqrt{10}}{5} & \frac{\sqrt{14}}{7} & -\frac{\sqrt{10}}{10} & \frac{\sqrt{70}}{70} \\ \frac{\sqrt{5}}{5} & -\frac{\sqrt{10}}{10} & -\frac{\sqrt{14}}{14} & \frac{\sqrt{10}}{5} & \frac{-2\sqrt{70}}{35} \\ \frac{\sqrt{5}}{5} & 0 & -\frac{\sqrt{14}}{7} & 0 & \frac{3\sqrt{70}}{35} \\ \frac{\sqrt{5}}{5} & \frac{\sqrt{10}}{10} & -\frac{\sqrt{14}}{14} & -\frac{\sqrt{10}}{5} & \frac{-2\sqrt{70}}{35} \\ \frac{\sqrt{5}}{5} & \frac{\sqrt{10}}{5} & \frac{\sqrt{14}}{7} & \frac{\sqrt{10}}{10} & \frac{\sqrt{70}}{70} \end{bmatrix} \begin{bmatrix} \sqrt{5} & 2\sqrt{5} & 2\sqrt{5} & \sqrt{5} & \frac{\sqrt{5}}{5} \\ 0 & \sqrt{10} & \frac{3\sqrt{10}}{2} & \frac{9\sqrt{10}}{10} & \frac{\sqrt{10}}{5} \\ 0 & 0 & \frac{\sqrt{14}}{2} & \frac{\sqrt{14}}{2} & \frac{\sqrt{14}}{7} \\ 0 & 0 & 0 & \frac{\sqrt{10}}{5} & \frac{\sqrt{10}}{10} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{70}}{70} \end{bmatrix}$$

11. Let V be a finite-dimensional, nonzero vector space over a field $\mathbb{F}$($\mathbb{F}$ could be either $\mathbb{R}$ or $\mathbb{C}$). An inner product on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle$ and has the following properties:
    - **Positivity.**
      $\langle v, v\rangle \geq 0$ for all $v \in V$;
    - **Definiteness.**
      $\langle v, v\rangle = 0$ if and only if $v = 0$;
    - **Additivity**
      $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$;
    - **Homogeneity**
      $\langle au, v\rangle = a\langle u, v\rangle$ for all $a \in F$ and all $u, v \in V$;
    - **Conjugate Symmetry**
      $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$;

    Recall that every real number is its complex conjugate, so if $\mathbb{F} = \mathbb{R}$, then the Conjugate Symmetry condition is automatic.

    Let $C[0,1]$ denote the real-valued continuous functions on [0,1]. Define an inner product by $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)dx$

a. Show that this is an inner product space.
SOLN:
Positivity:
$\langle f(x), f(x)\rangle = \int_0^1 \left(f(x)\right)^2 dx \geq 0$ since $\left(f(x)\right)^2 \geq 0$ for all x.
Definiteness:
$\langle f(x), f(x)\rangle = \int_0^1 \left(f(x)\right)^2 dx = 0 \Leftrightarrow f(x) = 0$
Additivity:
$\langle f(x) + g(x), h(x)\rangle = \langle u, w\rangle + \langle v, w\rangle$ follows because the integral of a sum is the sum of integrals.
Homogeneity:
$\langle au, v\rangle = a\langle u, v\rangle \Leftrightarrow$ you can factor a constant out of an integrand, which you can because of the distributive property for multiplication over addition.
Conjugate Symmetry
$\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$.
Soln: We're just working over the reals so $v = 0$ and there are no worries.

b. Apply the Gram-Schmidt method to get an orthonormal basis for subspace spanned by $\{1, \sin(x), \cos(x)\}$
SOLN:
A = a = 1; $B = \sin(x) - \frac{\int_0^1 \sin x dx}{\int_0^1 dx} = \sin x + \cos 1 - 1$

$$C = \cos x - \frac{\int_0^1 \cos x\, dx}{\int_0^1 dx} \cdot 1 - \frac{\int_0^1 (\sin x + \cos 1 - 1)\cos x\, dx}{\int_0^1 (\sin x + \cos 1 - 1)^2 dx}(\sin x + \cos 1 - 1)$$

$$= \cos x - \sin 1 - \frac{\frac{1}{2}\sin^2 1 + \sin 1 \cos 1 - \sin 1}{-\frac{1}{2} + 2\cos 1 - \frac{1}{2}\sin 1 \cos 1 - \cos^2 1}(\sin x + \cos 1 - 1)$$

I don't see a great simplification for this, so I guess that's about it? But need to check that these are orthogonal.
$\langle 1, \sin x + \cos 1 - 1\rangle = \int_0^1 (\sin x + \cos 1 - 1) dx = 0$
Ruh roh….Resort to Mathematica:
Define $f(x), g(x)$:

In[1]:= $f(\text{x\_}) := 1$

In[2]:= $f(x)$

Out[2]= 1

In[3]:= $g(\text{x\_}) := \sin(x)$

In[4]:= $g(x)$

Out[4]= $\sin(x)$

In[5]:= $G(\text{x\_}) := g(x) - \int_0^1 g(x)\, dx$

In[6]:= $G(x)$

Out[6]= $\sin(x) - 1 + \cos(1)$

In[7]:= $h(\text{x\_}) := \cos(x)$

In[8]:= h [x]

Out[8]= $\cos(x)$

In[9]:= $H(\text{x\_}) := -\frac{G(x)\left(\int_0^1 G(x)\, h(x)\, dx\right)}{\int_0^1 G(x)^2\, dx} + h(x) - \int_0^1 h(x)\, dx$

In[10]:= $H(x)$

Out[10]= $\cos(x) - \frac{2\sin(1)\,(-2 + \sin(1) + 2\cos(1))\,(\sin(x) - 1 + \cos(1))}{-4 - \sin(2) + 8\cos(1) - 2\cos(2)} - \sin(1)$

Then check that they're orthogonal:

Integrate[f[x]*G[x],{x,0,1}]
0
Integrate[f[x]*H[x],{x,0,1}]
0
Integrate[G[x]*H[x],{x,0,1}]
0
Hooray!