

Write all responses on separate paper. Show your work for credit.

1. Consider  $A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 4 & 5 \\ 2 & 4 & 4 & 2 \end{bmatrix}$

- a. Find a basis for row space.

SOLN:

- b. Use the Gram-Schmidt method to find an orthogonal basis for the row space. Hint: use the vectors in  $\text{rref}(A)$  and start with the simplest vector.
- c. Find a basis for the null space.
- d. Find a basis for the column space.
- e. Find a basis for the left null space.
- f. Verify the Fundamental Theorem of Linear Algebra for this matrix (both parts).
2. With  $z = 1, 0, 3, 1$  at  $(x, y) = (0, 1), (1, 0), (-1, 0), (0, -1)$ , set up and solve the normal equations  $A^T A \hat{x} = A^T b$  for the least squares plane  $z = Ax + By + C$  and find the projection  $p = A \hat{x}$  and the error  $e = b - p$ .
3. The vectors  $(1, 2, 2, 0)$  and  $(0, 1, -1, 2)$  are orthogonal. Divide them by their lengths to find orthonormal vectors  $\vec{q}_1$  and  $\vec{q}_2$ . Put those into the columns of  $Q$  and multiply  $Q^T Q$  and  $Q Q^T$ .
4. Find orthogonal vectors  $A, B, C$ , by Gram-Schmidt from  $\vec{a} = (1, 1, 3, 5)$   $\vec{b} = (1, 1, 1, -1)$  and  $\vec{c} = (1, 1, -1, 1)$ .
5. If  $\vec{u}$  is a unit vector, then  $Q = I - 2\vec{u}\vec{u}^T$  is a reflection matrix. Find  $Q$  from  $\vec{u} = (3, 4)/5$ . Draw the reflection when  $Q$  multiplies  $(0, 1)$ .

## Math 2B – Linear Algebra – Test 2 – (in class makeup) Solutions

1. Consider  $A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 4 & 5 \\ 2 & 4 & 4 & 2 \end{bmatrix}$

a. Find a basis for row space.

$$\text{SOLN: } A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 4 & 5 \\ 2 & 4 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -3 & 0 & 3 \\ 0 & -3 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we could take either  $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \\ 5 \end{bmatrix} \right\}$  or  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$  as a basis for the row space.

b. Use the Gram-Schmidt method to find an orthogonal basis for the row space. Hint: use the vectors in  $\text{rref}(A)$  and start with the simplest vector.

$$\text{ANS: Using the second pair, } A = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} \text{ and } B = b - \frac{A^T b}{A^T A} A = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{3}{14} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 3 \\ 14 \\ 6 \\ -5 \end{bmatrix}$$

c. Find a basis for the null space.

$$\text{SOLN: } \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \text{ basis} = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

d. Find a basis for the column space.

$$\text{SOLN: Since the pivots of } \text{rref}(A) \text{ are in the first two columns, } \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \right\}.$$

e. Find a basis for the left null space.

$$\text{SOLN: } A^T = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 2 & 4 & 8 \\ 1 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \\ 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ whence the left null space is spanned}$$

by  $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$

f. Verify the Fundamental Theorem of Linear Algebra for this matrix (both parts).

SOLN:  $\text{rank} = 2$ ,  $m = 3$ ,  $n = 4$  and, sure enough, the dimension of the null space is  $n - r = 4 - 2 = 2$  and the dimension of the left null space is  $m - r = 3 - 2 = 1$ .

The null space is perpendicular to the row space as evidenced by  $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 4 & 5 \\ 2 & 4 & 4 & 2 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} =$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \text{ Also } A^T \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 2 & 4 & 8 \\ 1 & 5 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

2. With  $z = 1, 0, 3, 1$  at  $(x, y) = (0, 1), (1, 0), (-1, 0), (0, -1)$ , set up and solve the normal equations  $A^T A \hat{x} = A^T b$  for the least squares plane  $z = Ax + By + C$  and find the projection  $p = A \hat{x}$  and the error  $e = b - p$ .

SOLN: The system of equations is described in the matrix equation: 
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} C \\ B \\ A \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

If we were to row-reduce we would see the system is over-determined so there is no solution. Thus we seek the best approximate solution by projecting the right hand side onto the column space.

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}^{-1}$$

$$\text{So } \hat{x} = (A^T A)^{-1} A^T b = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

So the least square plane has the equation  $z = \frac{1}{2}x + \frac{5}{4}$  (parallel to the y-axis).

3. The vectors  $(1, 2, 2, 0)$  and  $(0, 1, -1, 2)$  are orthogonal. Divide them by their lengths to find orthonormal vectors  $\vec{q}_1$  and  $\vec{q}_2$ . Put those into the columns of  $Q$  and multiply  $Q^T Q$  and  $Q Q^T$ .

$$\text{SOLN: } \vec{q}_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} \Rightarrow Q = \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 2 & -1 \\ 0 & 2 \end{bmatrix} \Rightarrow Q^T Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } Q Q^T = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 & 0 \\ 2 & 5 & 3 & 2 \\ 2 & 3 & 5 & -2 \\ 0 & 2 & -2 & 4 \end{bmatrix}$$

4. Find orthogonal vectors  $A, B, C$ , by Gram-Schmidt from  $\vec{a} = (1, 1, 3, 5)$ ,  $\vec{b} = (1, 1, 1, -1)$  and  $\vec{c} = (1, 1, -1, 1)$ .

$$\text{SOLN: } A = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 5 \end{bmatrix}, B = b - \frac{A^T b}{A^T A} A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} - 0 \begin{bmatrix} 1 \\ 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{4}{36} \begin{bmatrix} 1 \\ 1 \\ 3 \\ 5 \end{bmatrix} - 0 = \frac{1}{9} \begin{bmatrix} 8 \\ 8 \\ -12 \\ 4 \end{bmatrix}$$

5. If  $\vec{u}$  is a unit vector, then  $Q = I - 2\vec{u}\vec{u}^T$  is a reflection matrix. Find  $Q$  from  $\vec{u} = (3, 4)/5$ . Draw the reflection when  $Q$  multiplies  $(0, 1)$ .

$$\text{SOLN: } Q = I - \frac{2}{25} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2}{25} \begin{bmatrix} 9 & 12 \\ 12 & 9 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 7 & -24 \\ -24 & 7 \end{bmatrix}$$

