Math 2B – Linear Algebra – Test 2 – (in class) S13 Name______________________________

Write all responses on separate paper. Show your work for credit.

1. Consider $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 5 & 7 \\ 2 & 2 & 7 & 13 \end{bmatrix}$
   a. Find a basis for row space.
   b. Find a basis for the null space.
   c. Find a basis for the column space.
   d. Find a basis for the left null space.
   e. Verify the Fundamental Theorem of Linear Algebra for this matrix (both parts).

2. With $b = 1,0,3,1$ at $t = 0,1,2,3$, set up and solve the normal equations $A^T A\hat{x} = A^T b$ for the best parabola and find the projection $p = A\hat{x}$ and the error $e = b - p$.

3. The vectors (1,1,3,5) and $(1,1,1,-1)$ are orthogonal. Divide them by their lengths to find orthonormal vectors $\overrightarrow{q_1}$ and $\overrightarrow{q_2}$. Put those into the columns of Q and multiply $Q^T Q$ and QQ^T.

4. Find orthogonal vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, by Gram-Schmidt from $\vec{a} = (1,2,2,4)$ $\vec{b} = (1,2,-2,-4)$ and $\vec{c} = (1,1,-1,1)$.

5. If $\vec{u}$ is a unit vector, then $Q = I - 2\vec{u}\vec{u}^T$ is a reflection matrix. Find Q from $\vec{u} = (\sqrt{3}, 1)/2$. Draw the reflection when Q multiplies (1,0).

Math 2B – Linear Algebra – Test 2 – (in class) Solutions S13

1. Consider $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 5 & 7 \\ 2 & 2 & 7 & 13 \end{bmatrix}$
   a. Find a basis for row space.
   $$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 5 & 7 \\ 2 & 2 & 7 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
   We could take either the first two rows of rref(A) or we could use the first and third rows of A as a basis for the row space. So, either $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \\ 7 \end{bmatrix} \right\}$ or any other pair of vector that span this hyperplane.
   b. Find a basis for the null space.
   SOLN:
   $$\text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
   So $N = \begin{bmatrix} -1 & 4 \\ 1 & 0 \\ 0 & -3 \\ 0 & 1 \end{bmatrix}$ and so a basis is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$.
   c. Find a basis for the column space.
   SOLN: Since the matrix is symmetric, the column space is the same as the Row space (see part a).
   d. Find a basis for the left null space.
   SOLN: Same two-fer. Lucky you and me! Since $A^T = A$, the left null space is the same as the null space.
   e. Verify the Fundamental Theorem of Linear Algebra for this matrix (both parts).
   Well, since the rank is 2, the nullity of the two null spaces is 2 and it all adds up.
   That the row space is orthogonal to the null space is in evidence with the fact that the dot product of every vector in a basis for the row space (column space) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$ with a vector in a basis for the null space (left null space) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$ is zero.

2. With $b = 1{,}0{,}3{,}1$ at $t = 0{,}1{,}2{,}3$, set up and solve the normal equations $A^T A\hat{x} = A^T b$ for the best parabola and find the projection $p = A\hat{x}$ and the error $e = b - p$.
SOLN: The system of equations is captured in the matrix equation

$$\begin{bmatrix}1 & 0 & 0\\1 & 1 & 1\\1 & 2 & 4\\1 & 3 & 9\end{bmatrix}\begin{bmatrix}c\\b\\a\end{bmatrix} = \begin{bmatrix}1\\0\\3\\1\end{bmatrix}$$

Multiplying both sides by A^T we get

$$\begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 2 & 3\\0 & 1 & 4 & 9\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\1 & 1 & 1\\1 & 2 & 4\\1 & 3 & 9\end{bmatrix}\begin{bmatrix}c\\b\\a\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 2 & 3\\0 & 1 & 4 & 9\end{bmatrix}\begin{bmatrix}1\\0\\3\\1\end{bmatrix}$$

$$\begin{bmatrix}4 & 6 & 14\\6 & 14 & 36\\14 & 36 & 98\end{bmatrix}\begin{bmatrix}c\\b\\a\end{bmatrix} = \begin{bmatrix}5\\9\\21\end{bmatrix}$$

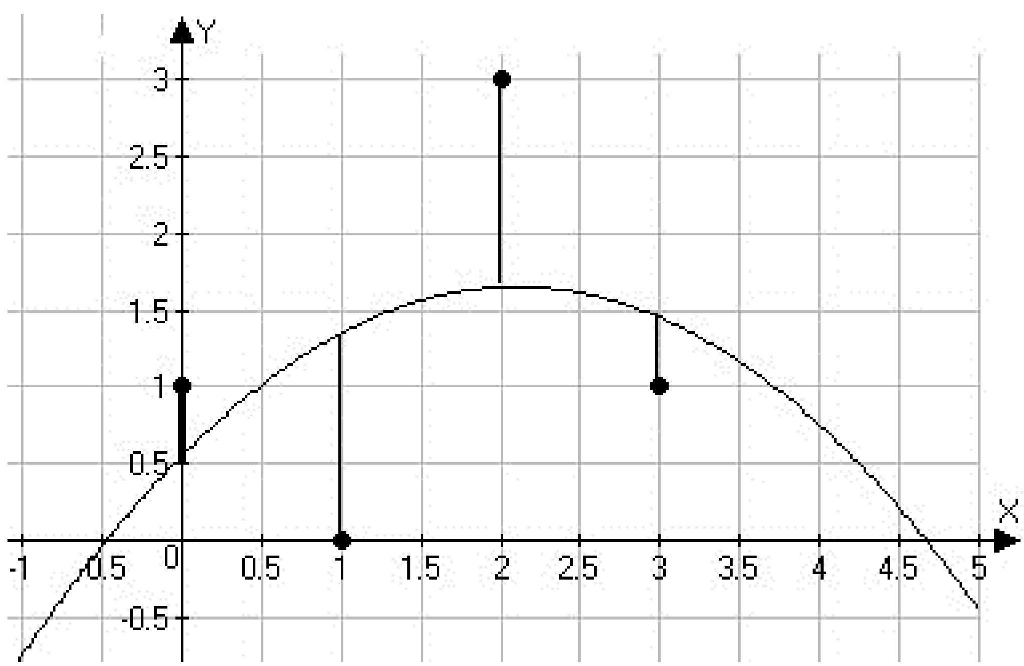
3. This system can be solved by doing row reduction on an augmented matrix:

$$\begin{bmatrix}4 & 6 & 14 & 5\\6 & 14 & 36 & 9\\14 & 36 & 98 & 21\end{bmatrix} \sim \begin{bmatrix}4 & 6 & 14 & 5\\0 & 5 & 15 & 3/2\\0 & 15 & 49 & 7/2\end{bmatrix} \sim \begin{bmatrix}4 & 6 & 14 & 5\\0 & 5 & 15 & 3/2\\0 & 0 & 4 & -1\end{bmatrix}$$

So a = – ¼ , which means the parabola opens downwards. Back-substituting we get $b = \frac{1}{5}\left(\frac{3}{2} + \frac{15}{4}\right) = \frac{21}{20}$ and $c = \frac{1}{4}\left(5 - 6 \cdot \frac{21}{20} + 14 \cdot \frac{1}{4}\right) = \frac{11}{20}$ so that the parabola is described by $y = -\frac{1}{4}t^2 + \frac{21}{20}t + \frac{11}{20} = -\frac{1}{4}\left(t - \frac{21}{10}\right)^2 + \frac{661}{400}$. $p = A\hat{x} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 1\\1 & 2 & 4\\1 & 3 & 9\end{bmatrix}\begin{bmatrix}11/20\\21/20\\-1/4\end{bmatrix} = \begin{bmatrix}11/20\\27/20\\33/20\\29/20\end{bmatrix}$

and the error is

$$b - p = \begin{bmatrix}1\\0\\3\\1\end{bmatrix} - \begin{bmatrix}11/20\\27/20\\33/20\\29/20\end{bmatrix} = \begin{bmatrix}0.45\\-1.35\\1.35\\-0.45\end{bmatrix}$$



Here is the same problem with a slightly different set of data ($y(2) = 2$, instead of 3).
The system of equations is captured in the matrix equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Multiplying both sides by A^T we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 17 \end{bmatrix}$$

This system can be solved by doing row reduction on an augmented matrix:

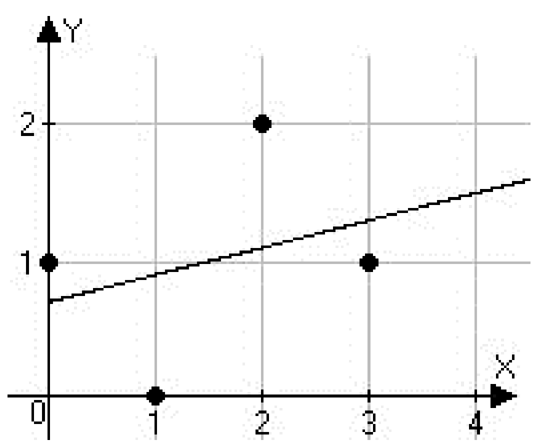
$$\begin{bmatrix} 4 & 6 & 14 & 4 \\ 6 & 14 & 36 & 7 \\ 14 & 36 & 98 & 17 \end{bmatrix} \sim \begin{bmatrix} 4 & 6 & 14 & 4 \\ 0 & 5 & 15 & 1 \\ 0 & 15 & 49 & 3 \end{bmatrix} \sim \begin{bmatrix} 4 & 6 & 14 & 4 \\ 0 & 5 & 15 & 1 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

So a = 0, which means it's a degenerate parabola. Thus b = 1/5 is the slope of the line and c = 7/10 is the intercept and $p = A\hat{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} 7/10 \\ 1/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 7/10 \\ 9/10 \\ 11/10 \\ 13/10 \end{bmatrix}$

and the error is

$$b - p = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{7}{10} \\ \frac{9}{10} \\ \frac{11}{10} \\ \frac{13}{10} \end{bmatrix} = \begin{bmatrix} 0.3 \\ -0.9 \\ 0.9 \\ -0.3 \end{bmatrix}$$

And the picture is



4. The vectors $(1,1,3,5)$ and $(1,1,1,-1)$ are orthogonal. Divide them by their lengths to find orthonormal vectors $\overrightarrow{q_1}$ and $\overrightarrow{q_2}$. Put those into the columns of Q and multiply Q^TQ and QQ^T.
SOLN:
The first vector has length $\sqrt{1^2+1^2+3^2+5^2}=6$ and the second vector, $\sqrt{1^2+1^2+1^2+(-1)^2}=2$ so that $Q^TQ=\frac{1}{36}\begin{bmatrix}1&1&3&5\\3&3&3&-3\end{bmatrix}\begin{bmatrix}1&3\\1&3\\3&3\\5&-3\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$$QQ^T=\frac{1}{36}\begin{bmatrix}1&3\\1&3\\3&3\\5&-3\end{bmatrix}\begin{bmatrix}1&1&3&5\\3&3&3&-3\end{bmatrix}=\frac{1}{18}\begin{bmatrix}5&5&6&-2\\5&5&6&-2\\6&6&9&3\\-2&-2&3&17\end{bmatrix}$$

That was easy.

5. Find orthogonal vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, by Gram-Schmidt from $\vec{a}=(1,2,2,4)$ $\vec{b}=(1,2,-2,-4)$ and $\vec{c}=(1,1,-1,1)$.

SOLN: $A=\begin{bmatrix}1\\2\\2\\4\end{bmatrix}$, $B=\begin{bmatrix}1\\2\\-2\\-4\end{bmatrix}+\frac{15}{25}\begin{bmatrix}1\\2\\2\\4\end{bmatrix}=\frac{4}{5}\begin{bmatrix}2\\4\\-1\\-2\end{bmatrix}$, $C=\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}-\frac{5}{25}\begin{bmatrix}1\\2\\2\\4\end{bmatrix}-\frac{4}{16}\frac{4}{5}\begin{bmatrix}2\\4\\-1\\2\end{bmatrix}=\frac{1}{5}\begin{bmatrix}2\\-1\\-2\\1\end{bmatrix}$

Normalizing, we get

$$\vec{q}_1=\frac{1}{5}\begin{bmatrix}1\\2\\2\\4\end{bmatrix},\vec{q}_2=\frac{1}{5}\begin{bmatrix}2\\4\\-1\\-2\end{bmatrix},\vec{q}_3=\frac{1}{5}\begin{bmatrix}2\\-1\\-2\\1\end{bmatrix}$$

6. If $\vec{u}$ is a unit vector, then $Q=I-2\vec{u}\vec{u}^T$ is a reflection matrix. Find Q from $\vec{u}=(\sqrt{3},1)/2$. Draw the reflection when Q multiplies $(1,0)$.

SOLN: $\vec{u}\vec{u}^T=\frac{1}{4}\begin{bmatrix}3&\sqrt{3}\\\sqrt{3}&1\end{bmatrix}$ so $Q=I-2\vec{u}\vec{u}^T=$
$\begin{bmatrix}1&0\\0&1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}3&\sqrt{3}\\\sqrt{3}&1\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}1&\sqrt{3}\\\sqrt{3}&-1\end{bmatrix}$
Thus, $(1,0)$ gets mapped to
$Q\begin{bmatrix}1\\0\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}1&\sqrt{3}\\\sqrt{3}&-1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}1\\\sqrt{3}\end{bmatrix}$