Math 2B – Linear Algebra – Test 1 – S13 Name____________________________

Write all responses on separate paper. Show your work for credit.

1. Find a unit vector $\hat{u}$ parallel to [3,4] and a unit vector $\hat{w}$ in the parallel to [5,12] and then compute $(\hat{u}+\hat{w})\cdot(\hat{u}-\hat{w})$

2. Find a combination $x_1\overrightarrow{w_1}+x_2\overrightarrow{w_2}+x_3\overrightarrow{w_3}$ that gives the zero vector where
$\overrightarrow{w_1}=\begin{bmatrix}1\\4\\7\end{bmatrix}$ $\overrightarrow{w_2}=\begin{bmatrix}2\\5\\8\end{bmatrix}$ $\overrightarrow{w_3}=\begin{bmatrix}3\\6\\9\end{bmatrix}$

3. Draw the row and column pictures for the equations $x+2y=5$ and $x-y=-1$

4. Show that the three vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ that make up the rows of $C=\begin{bmatrix}\vec{r}_1\\\vec{r}_2\\\vec{r}_3\end{bmatrix}=\begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{bmatrix}$ are coplanar, that is, they all lie in one plane. Hint: Show there is a non-trivial solution to $C\vec{x}=\vec{0}$.

5. What 2 by 2 matrix R **rotates every vector through 30°? Find the inverse of R and show that $R^{-1}R=I_2$.**

6. Consider the system of equations

$$\begin{aligned}2x+y\ \ &=29\\ x+2y+z&=65\\ y+2z&=51\end{aligned}$$

   a. Reduce this system to upper triangular form by two row operations.
   b. Write this system as a matrix equation $A\vec{x}=\vec{b}$ and find the $A=LU$ factorization of A.

7. Consider the matrices,

$$E=\begin{bmatrix}1&0&0\\a&1&0\\b&0&1\end{bmatrix},\qquad F=\begin{bmatrix}1&0&0\\0&1&0\\0&c&1\end{bmatrix}$$

   a. Compute EF and FE:
   b. Compute E^3 and F^3.
   c. Find formulas for E^n and F^n.

8. Let $A=\begin{bmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}$ compute the inverse matrix of A^4 and use it to solve $A^4\vec{x}=\vec{0}$.

9. Find the LDU factorization of $A=\begin{bmatrix}2&6&4\\2&7&6\\4&15&17\end{bmatrix}$.

Math 2B – Linear Algebra – Test 1 Solutions – S13

1. Find a unit vector $\hat{u}$ parallel to [3,4] and a unit vector $\hat{w}$ in the parallel to [5,12] and then compute $(\hat{u}+\hat{w})\cdot(\hat{u}-\hat{w})$
   SOLN: $\hat{u}=\left[\frac{3}{5},\frac{4}{5}\right]$ and $\hat{w}=\left[\frac{5}{13},\frac{12}{13}\right]$. So $(\hat{u}+\hat{w})\cdot(\hat{u}-\hat{w})=\left[\frac{64}{65},\frac{112}{65}\right]\cdot\left[\frac{14}{65},-\frac{8}{65}\right]=\left[\frac{896}{65^2}-\frac{896}{65^2}\right]=0.$
   More simply, since unit vectors form the sides of a rhombus, and the diagonals are perpendicular, this is 0.
2. Find a combination $x_1\overrightarrow{w_1}+x_2\overrightarrow{w_2}+x_3\overrightarrow{w_3}$ that gives the zero vector where

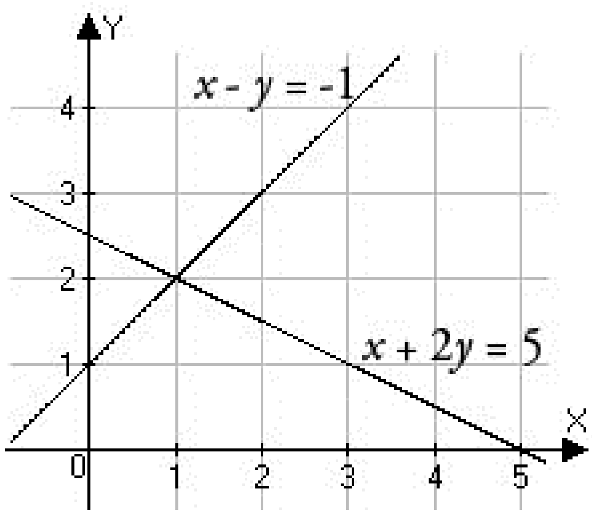
   $$\overrightarrow{w_1}=\begin{bmatrix}1\\4\\7\end{bmatrix}\qquad \overrightarrow{w_2}=\begin{bmatrix}2\\5\\8\end{bmatrix}\qquad \overrightarrow{w_3}=\begin{bmatrix}3\\6\\9\end{bmatrix}$$

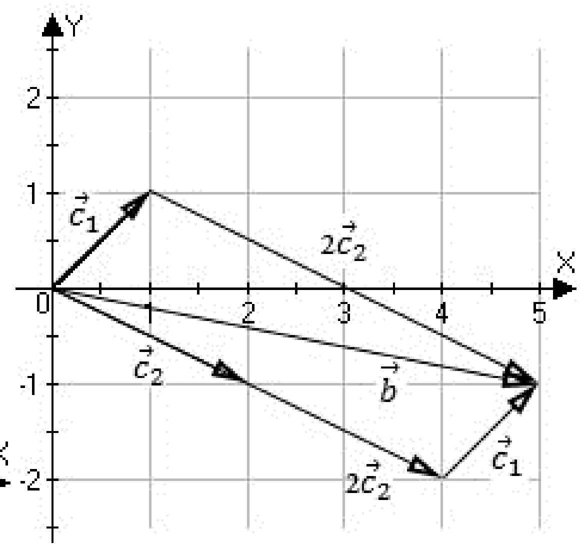
   SOLN: This problem asks for a linear combination of the vectors that is the zero vector. This is equivalent to solving $\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ The augmented matrix for this system is

   $$\begin{bmatrix}1&2&3&0\\4&5&6&0\\7&8&9&0\end{bmatrix}\sim\begin{bmatrix}1&2&3&0\\0&-3&-6&0\\0&-6&-12&0\end{bmatrix}\sim\begin{bmatrix}1&2&3&0\\0&1&2&0\\0&0&0&0\end{bmatrix}$$ so all solutions are of the form $\vec{x}=t\begin{bmatrix}1\\-2\\1\end{bmatrix}$

   Taking t = 2, for instance, we get $2\overrightarrow{w_1}-4\overrightarrow{w_2}+2\overrightarrow{w_3}=\vec{0}$.

3. Draw the row and column pictures for the equations $x+2y=5$ and $x-y=-1$
   SOLN: The row picture is just the graph of the two lines showing their point of intersection, , as the solution to the system. The column vectors are $\vec{c}_1=\begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{c}_2=\begin{bmatrix}2\\-1\end{bmatrix}$ and the column picture shows how multiples of these vectors combine to form the RHS column, $\vec{b}=\begin{bmatrix}5\\-1\end{bmatrix}$. Evidently, $\vec{c}_1+2\vec{c}_2=\vec{b}$





4. Show that the three vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ that make up the rows of $C=\begin{bmatrix}\vec{r}_1\\\vec{r}_2\\\vec{r}_3\end{bmatrix}=\begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{bmatrix}$
   are coplanar, that is, they all lie in one plane. Hint: Show there is a non-trivial solution to $C\vec{x}=\vec{0}$.
   ANS: Since $\vec{r}_3=-\vec{r}_2-\vec{r}_1$, $\vec{r}_3$ is a linear combination of $\vec{r}_2$ and $\vec{r}_1$, which means that $\vec{r}_3$ is in the plane containing $\vec{r}_2$ and $\vec{r}_1$.

5. What 2 by 2 matrix R rotates every vector through 30°? Find the inverse of R and show that $R^{-1}R = I_2$.

SOLN: In general, a 2 by 2 rotation matrix has the form $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$, but if you forget that, look at what should happen to the unit vectors: $R\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\sqrt{3}/2\\1/2\end{bmatrix}$ and $R\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1/2\\\sqrt{3}/2\end{bmatrix}$, so, by inspection,

$$R = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

6. Consider the system of equations

$$\begin{aligned} 2x + y \quad &= 29 \\ x + 2y + z &= 65 \\ y + 2z &= 51 \end{aligned}$$

a. Reduce this system to upper triangular form by two row operations.

SOLN: The augmented matrix is $\begin{bmatrix} 2 & 1 & 0 & 29 \\ 1 & 2 & 1 & 65 \\ 0 & 1 & 2 & 51 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 & 29 \\ 0 & 3/2 & 1 & 101/2 \\ 0 & 1 & 2 & 51 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 & 29 \\ 0 & 3/2 & 1 & 101/2 \\ 0 & 0 & 4/3 & 52/3 \end{bmatrix}$

b. Write this system as a matrix equation $A\vec{x} = \vec{b}$ and find the $A = LU$ factorization of A.

SOLN: You could go one further for the *LDU* version:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 29 \\ 65 \\ 51 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/3 & 2/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

7. Consider the matrices,

$$E = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

a. Compute EF and FE:

SOLN: $EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$

and $FE = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ ca + b & c & 1 \end{bmatrix}$

b. Compute E^3 and F^3.

SOLN:

$$E^3 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3a & 1 & 0 \\ 3b & 0 & 1 \end{bmatrix}$$

$$F^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3c & 1 \end{bmatrix}$$

c. Find formulas for E^n and F^n.

SOLN: Claim:

$$E^n = \begin{bmatrix} 1 & 0 & 0 \\ na & 1 & 0 \\ nb & 0 & 1 \end{bmatrix}$$

Proof by induction: We've shown the formula works for *n* = 2 and 3. Assume it works for *n* – 1, Then

$$E^n = \begin{bmatrix} 1 & 0 & 0 \\ (n-1)a & 1 & 0 \\ (n-1)b & 0 & 1 \end{bmatrix} E = \begin{bmatrix} 1 & 0 & 0 \\ na & 1 & 0 \\ nb & 0 & 1 \end{bmatrix}$$

Similarly,

$$F^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & (n-1)c & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & nc & 1 \end{bmatrix}$$

8. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ compute the inverse matrix of A^4 and use it to solve $A^4\vec{x} = \vec{0}$.

SOLN: $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

So $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ so $A^{-2} = (A^{-1})^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$

Thus $(A^4)^{-1} = A^{-4} = (A^{-2})^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 10 & 4 & 1 & 0 \\ 20 & 10 & 4 & 1 \end{bmatrix}$

And, well, of course, the only solution is $\vec{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 10 & 4 & 1 & 0 \\ 20 & 10 & 4 & 1 \end{bmatrix} \vec{0} = \vec{0}$

9. Find the LDU factorization of $A = \begin{bmatrix} 2 & 6 & 4 \\ 2 & 7 & 6 \\ 4 & 15 & 17 \end{bmatrix}$.

SOLN: $A = \begin{bmatrix} 2 & 6 & 4 \\ 2 & 7 & 6 \\ 4 & 15 & 17 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 \\ 0 & 1 & 2 \\ 4 & 15 & 17 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 \\ 0 & 1 & 2 \\ 0 & 3 & 9 \end{bmatrix} =$

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$