

Write all responses on separate paper. Show your work for credit.

1. True or false, with reason if true and counterexample if false:
  - a. Every invertible matrix  $A$  can be factored as  $A = LU$  where  $L$  is lower triangular and has all 1's on its diagonal and  $U$  is upper triangular with the pivots on its diagonal. Before confirming your answer, consider  $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ . What's going on?
  - b. A matrix  $A \in \mathbb{R}^{n \times n}$  can be factored as  $A = QR$ , where  $Q$  is orthonormal and  $R$  is upper triangular, only if  $A$  is non-singular.
  - c. Every invertible matrix can be diagonalized.
  - d. Every diagonal matrix can be inverted.
  - e. Exchanging the rows of a  $2 \times 2$  matrix reverses the signs of its eigenvalues.

2. Consider  $A = \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}$ .

- a. Factor  $A = PLDU$  as a product elementary matrices where  $P$  is a permutation matrix,  $L$  is lower triangular with 1's on the diagonal,  $U$  is upper triangular with 1's on the diagonal and  $D$  is a diagonal matrix (not that  $I$  is such a matrix). Use these to interpret the transformation  $y = Ax$  geometrically as some combination of dilations/contractions, shears and/or reflections.
- b. Factor  $A$  as a product of an orthogonal matrix  $Q$  and an upper triangular matrix  $R$ . Hint: First use gram-schmidt to find  $Q$  and then compute  $R = Q^{-1}A$ .

3. Consider  $\vec{b} = (0, 3, 0)$ .

- a. Project  $\vec{b}$  onto each of the orthonormal vectors  $\vec{u}_1 = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$  and  $\vec{u}_2 = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$  and find its projection onto the plane of  $\vec{u}_1$  and  $\vec{u}_2$ .
- b. Project  $\vec{b}$  onto  $\vec{u}_3 = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$  and add this projection to the two projections you found in part (a). How do you interpret the result?

4. If  $\mathbf{u}$  is a unit vector, show that  $Q = I - 2\mathbf{u}\mathbf{u}^T$  is an orthogonal matrix. Compute  $Q$  when  $\mathbf{u} = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ .

5. Let  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

- Draw a diagram illustrating the column space of  $A$ . Hint draw a vector in the  $xz$ -plane and a vector in the  $yz$ -plane and then a third vector from the endpoint of one to the endpoint of the other...this is a triangle contained in the plane of the column space.
- Draw a diagram illustrating the row space of  $A$ . Hint: Draw a basis vector in the  $xy$ -plane and a basis vector in the  $yz$ -plane and then a vector from the endpoint of one to the endpoint of the other...this is a triangle contained in the plane of the row space.
- Draw a diagram illustrating the null space of  $A$ .
- Draw a diagram illustrating the left null space of  $A$ .
- Let  $b = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ . Find the complete solution to  $Ax = b$ .

6. Find the eigenvalues and eigenvectors and a diagonalizing matrix for  $A = \begin{pmatrix} 7 & 2 \\ -15 & -4 \end{pmatrix}$ .

7. Find the determinants of  $A$  and  $A^{-1}$  if  $A = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} P^{-1}$ .

8. Suppose that  $A$  has eigenvalues 0 and 1, corresponding to the eigenvectors  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ .

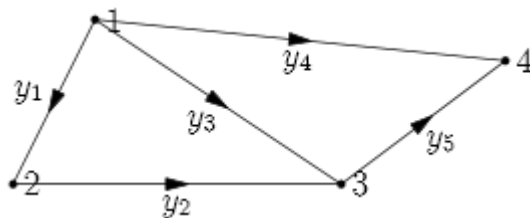
- How can you tell in advance (without actually computing  $A$ ) that  $A$  is symmetric?
- What are the trace and determinant of  $A$ ?
- What are the eigenvalues, eigenvectors and determinant of  $A^2$ ?

9. Find  $\lim_{k \rightarrow \infty} \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}^k \begin{bmatrix} a \\ b \end{bmatrix}$

10. Consider the network shown at right.

Whose incidence matrix is

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$



- Find the incidence matrix  $A$  for this network.
- Compute  $A^T A$  and find its eigenvalues.
- Compute the eigenvectors for  $A^T A$ .

Math 2B—Spring '13—Final Exam Solutions

1. True or false, with reason if true and counterexample if false:

- a. Every invertible matrix  $A$  can be factored as  $A = LU$  where  $L$  is lower triangular and has all 1's on its diagonal and  $U$  is upper triangular with the pivots on its diagonal. Before

confirming your answer, consider  $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ . What's going on?

SOLN: This is almost true, with the exception that sometimes a permutation matrix is required. The proposed example is factored like so:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = PLU$$

- b. A matrix  $A \in \mathbb{R}^{n \times n}$  can be factored as  $A = QR$ , where  $Q$  is orthonormal and  $R$  is upper triangular, only if  $A$  is non-singular.

SOLN: No. Here's a quick counterexample:  $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ .

- c. Every invertible matrix can be diagonalized.

SOLN: No. It could be the matrix has a repeated eigenvalue with only one eigenvector."

For example if  $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$  then the characteristic equation is  $(\lambda - 3)^2 = 0$  so there is only one eigenvector  $v = [1, 0]$ , so you can't diagonalize.

- d. Every diagonal matrix can be inverted.

SOLN: No. If there's a zero on the diagonal then the matrix is singular.

- e. Exchanging the rows of a  $2 \times 2$  matrix reverses the signs of its eigenvalues.

SOLN: No. Suppose the rows are the same. Then swapping rows won't change anything.

2. Consider  $A = \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$ .

- a. Factor  $A = PLDU$  as a product elementary matrices where  $P$  is a permutation matrix,  $L$  is lower triangular with 1's on the diagonal,  $U$  is upper triangular with 1's on the diagonal and  $I$  is such a matrix). Use these to interpret the transformation  $y = Ax$  geometrically as some combination of dilations/contractions, shears and/or reflections.

$$\text{SOLN: } A = \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = LDU.$$

This first shears by a factor of 4 parallel to the  $x$ -axis. Then stretch by a factor of 4 in the  $y$  direction while reflecting across the  $x$ -axis. Finally, a shear by a factor of 1 parallel to the  $x$ -axis.

- b. Factor  $A$  as a product of an orthogonal matrix  $Q$  and an upper triangular matrix  $R$ . Hint: First use gram-schmidt to find  $Q$  and then compute  $R = Q^{-1}A$ .

SOLN: Using the gram-schmidt formula (not really needed here, but it shows good

practice)  $B = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ . So  $Q = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$  and

$$R = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & -2\sqrt{2} \end{bmatrix}. \text{ So } \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & -2\sqrt{2} \end{bmatrix}$$

3. Consider  $\vec{b} = (0,3,0)$

a. Project  $\vec{b}$  onto each of the orthonormal vectors  $\vec{u}_1 = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$  and  $\vec{u}_2 = 2\left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$  and find its projection onto the plane of  $\vec{u}_1$  and  $\vec{u}_2$ .

SOLN: The projection of  $\vec{b}$  onto  $\vec{u}_1$  is  $\frac{\vec{b} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 = 2\left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$  and the projection of  $\vec{b}$  onto  $\vec{u}_2$  is  $\frac{\vec{b} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_1 = 2\left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ . The projection of  $\vec{b}$  onto the plane of  $\vec{u}_1$  and  $\vec{u}_2$  is the sum of these vectors:  $2\left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right) + 2\left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \left(\frac{2}{3}, \frac{8}{3}, \frac{2}{3}\right)$ .

b. Project  $\vec{b}$  onto  $\vec{u}_3 = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$  and add this projection to the two projections you found in part (a). How do you interpret the result?

SOLN:  $\frac{\vec{b} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} \vec{u}_3 = -1\left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) = \left(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \vec{b} - \left(\frac{2}{3}, \frac{8}{3}, \frac{2}{3}\right) = \left(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ .

4. If  $\mathbf{u}$  is a unit vector, show that  $Q = I - \mathbf{u}\mathbf{u}^T$  is an orthogonal matrix. Compute  $Q$  when  $\mathbf{u} = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ .

SOLN: My apologies... There was a typo in the problem statement which I didn't notice and wasn't called during the exam. That's unfortunate. The orthogonal matrix is  $Q = I - 2\mathbf{u}\mathbf{u}^T$ .  $Q^T Q = (I - 2\mathbf{u}\mathbf{u}^T)^T (I - 2\mathbf{u}\mathbf{u}^T) = (I - 2\mathbf{u}^T \mathbf{u})(I - 2\mathbf{u}\mathbf{u}^T) = I - 2I - 2I + 4I = I$  shows the intended matrix is orthonormal, however, the given matrix has

$$Q^T Q = (I - \mathbf{u}\mathbf{u}^T)^T (I - \mathbf{u}\mathbf{u}^T) = I - I - I + I = 0$$

indicating that it's a projection onto the null space of something maybe, but it's not an orthonormal matrix!

$$\text{When } \mathbf{u} = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right), I - \mathbf{u}\mathbf{u}^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} =$$

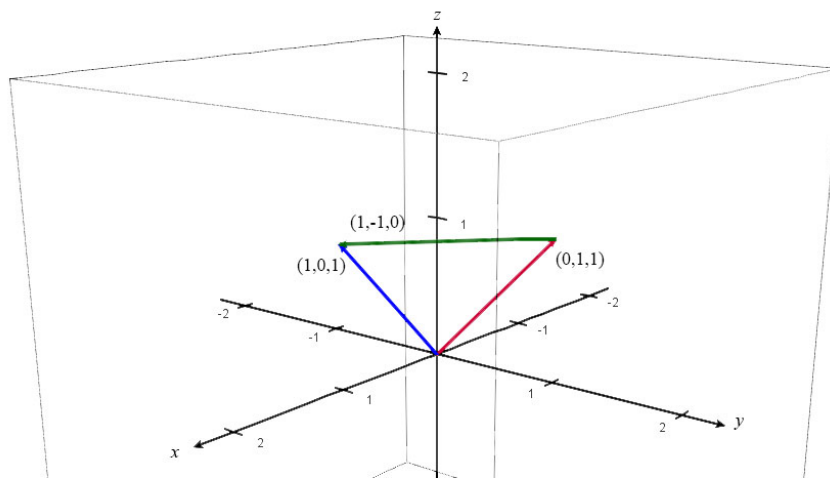
$$\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & -1 & 1 & 1 \\ -1 & 3 & 1 & 1 \\ 1 & 1 & 3 & -1 \\ 1 & 1 & -1 & 3 \end{bmatrix}. \text{ Note this is not orthogonal. Look at}$$

$$I - 2\mathbf{u}\mathbf{u}^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

That's orthonormal.

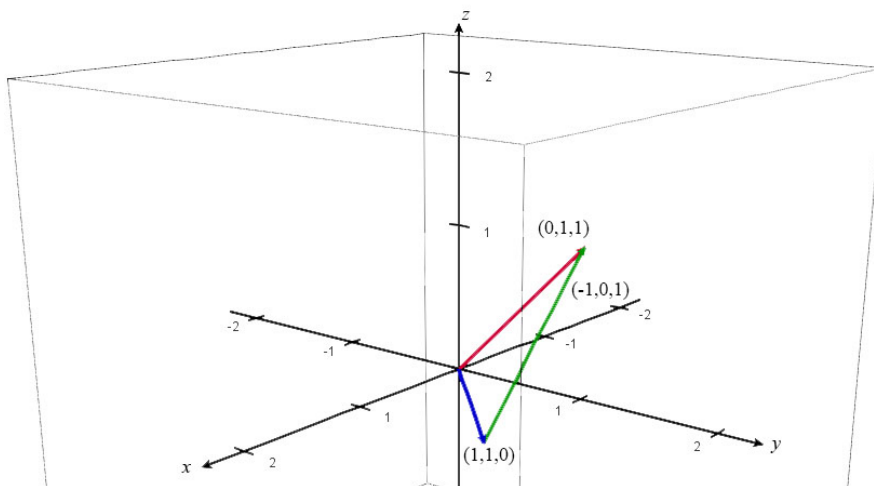
5. Let  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

- a. Draw a diagram illustrating the column space of  $A$ . Hint draw a vector in the  $xz$ -plane and a vector in the  $yz$ -plane and then a third vector from the endpoint of one to the endpoint of the other...this is a triangle contained in the plane of the column space.



SOLN->

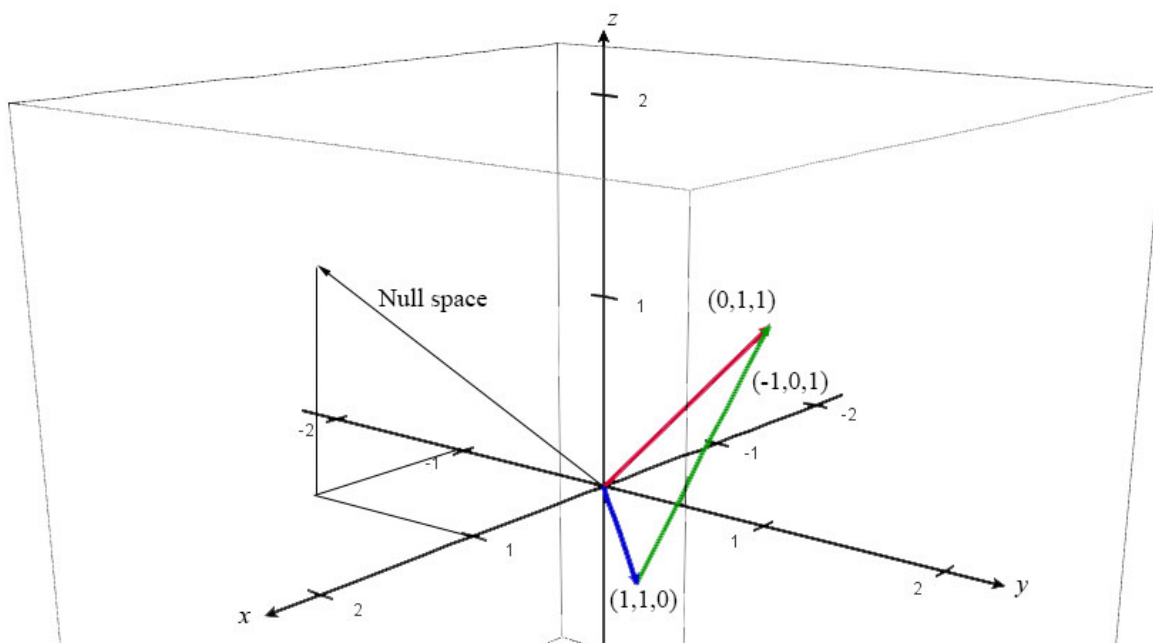
- b. Draw a diagram illustrating the row space of  $A$ . Hint: Draw a basis vector in the  $xy$ -plane and a basis vector in the  $yz$ -plane and then a vector from the endpoint of one to the endpoint of the other...this is a triangle contained in the plane of the row space. SOLN:



- c. Draw a diagram illustrating the null space of  $A$ .

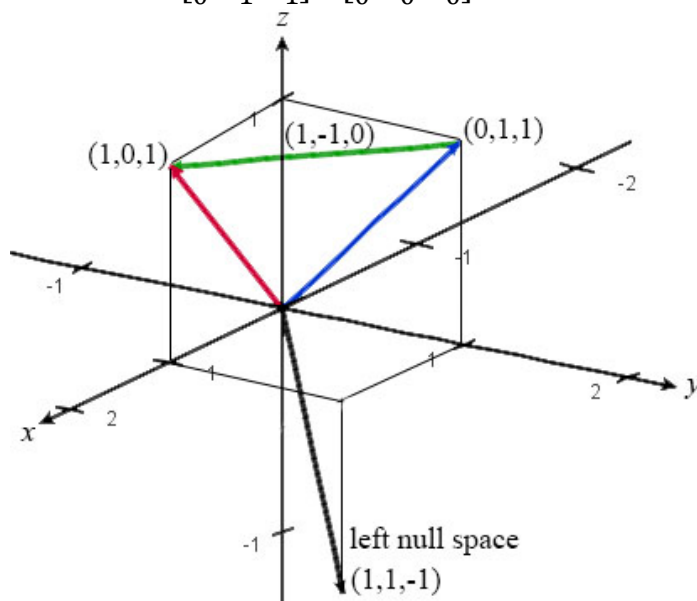
SOLN:  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  and so the null space is the span of the vector  $(1, -1, 1)$ .

Note that the  $N(A)$  is a line orthogonal to the plane of the row space,  $C(A^T)$ .



d. Draw a diagram illustrating the left null space of  $A$ .

SOLN:  $A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  and so the left null space is the span of  $(1, 1, -1)$ .



Note that the left null space  $N(A^T)$  is orthogonal to the column space  $C(A)$ .

e. Let  $b = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ . Find the complete solution to  $Ax = b$ .

SOLN:  $Ax = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \Leftrightarrow x = x_p + tx_n = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

6. Find the eigenvalues and eigenvectors and a diagonalizing matrix for  $A = \begin{bmatrix} 7 & 2 \\ -15 & -4 \end{bmatrix}$ .

SOLN:  $|A - \lambda I| = (7 - \lambda)(-4 - \lambda) + 30 = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) \Rightarrow$  the eigenvalues are 2 and 1. For  $\lambda = 2$  the eigenvector is  $(2, -5)$  and for  $\lambda = 1$  the eigenvector is  $(-1, 3)$ . Thus  $S = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ . To be sure,  $S\Lambda S^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ -15 & -4 \end{bmatrix}$  Nifty.

7. Find the determinants of  $A$  and  $A^{-1}$  if  $A = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} P^{-1}$ .

SOLN:  $|A| = \lambda_1^2$  and  $|A^{-1}| = \frac{1}{\lambda_1^2}$

8. Suppose that  $A$  has eigenvalues 0 and 1, corresponding to the eigenvectors  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ .

- a. How can you tell in advance (without actually computing  $A$ ) that  $A$  is symmetric?

SOLN: The diagonalizing matrix is (anti) symmetric. That is,

$$A = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

is a product of (anti) symmetric matrices, and so must itself be symmetric. Also, note that Further, note that it is \*not\* sufficient for  $A$  to have different eigenvalues/vectors. For

example,  $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$  is not symmetric and yet has different eigenvectors and eigenvalues.

- b. What are the trace and determinant of  $A$ ?

**SOLN:** Since  $A$  has a zero eigenvalue, the determinant is zero. The trace is the sum of eigenvalues = 1.

- a. What are the eigenvectors and eigenvalues of  $A^2$ ?

**SOLN:**

$$A^2 = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix},$$

so it has the same eigenvalues and eigenvectors.

9. Find  $\lim_{k \rightarrow \infty} \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}^k \begin{bmatrix} a \\ b \end{bmatrix}$

SOLN: The characteristic equation:

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 0.4 - \lambda & 0.3 \\ 0.6 & 0.7 - \lambda \end{vmatrix} = \left(\frac{2}{5} - \lambda\right) \left(\frac{7}{10} - \lambda\right) - \frac{9}{50} = \lambda^2 - \frac{11}{10}\lambda - \frac{1}{10} \\ &= (\lambda - 1) \left(\lambda - \frac{1}{10}\right) \end{aligned}$$

shows eigenvalues are 1 and 1/10 with corresponding eigenvectors  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

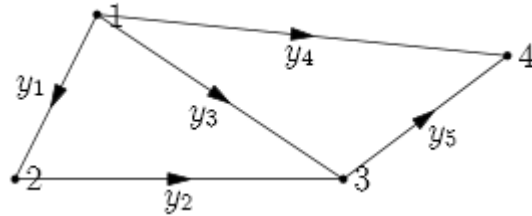
Thus

$$\begin{aligned} \lim_{k \rightarrow \infty} \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}^k \begin{bmatrix} a \\ b \end{bmatrix} &= \lim_{k \rightarrow \infty} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}^k \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \end{bmatrix} = \\ \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{3} \begin{bmatrix} a + b \\ 2a + 2b \end{bmatrix} \end{aligned}$$

10. Consider the network shown at right.

Whose incidence matrix is

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$



a. Compute  $A^T A$  and find its eigenvalues.

$$A^T A = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3 - \lambda & -1 & -1 & -1 \\ -1 & 2 - \lambda & -1 & 0 \\ -1 & -1 & 3 - \lambda & -1 \\ -1 & 0 & -1 & 2 - \lambda \end{vmatrix}$$

$$\begin{aligned} &= \begin{vmatrix} -1 & 2 - \lambda & -1 \\ -1 & -1 & 3 - \lambda \\ -1 & 0 & -1 \end{vmatrix} + \begin{vmatrix} 3 - \lambda & -1 & -1 \\ -1 & 2 - \lambda & -1 \\ -1 & 0 & -1 \end{vmatrix} + (2 - \lambda) \begin{vmatrix} 3 - \lambda & -1 & -1 \\ -1 & -1 & 3 - \lambda \end{vmatrix} \\ &= -1 - (2 - \lambda)(3 - \lambda) + 1 - (2 - \lambda) - (2 - \lambda)(3 - \lambda) - 1 - (2 - \lambda) \\ &\quad + 1 + (2 - \lambda)[(3 - \lambda)(2 - \lambda)(3 - \lambda) - 2 - (2 - \lambda) - 2(3 - \lambda)] \\ &= -(2 - \lambda)(8 - 2\lambda) + (2 - \lambda)^2(3 - \lambda)^2 - (2 - \lambda)(10 - 3\lambda) \\ &= (2 - \lambda)^2(3 - \lambda)^2 - (2 - \lambda)(18 - 5\lambda) \\ &= (2 - \lambda)[(2 - \lambda)(9 - 6\lambda + \lambda^2) - 18 + 5\lambda] \\ &= (2 - \lambda)(-\lambda^3 + 8\lambda^2 - 16\lambda) = \lambda(\lambda - 4)^2(\lambda - 2) \end{aligned}$$

So the eigenvalues are  $\lambda = 4$  (twice),  $\lambda = 2$  and  $\lambda = 0$ .

b. Compute the eigenvectors for  $A^T A$ .

SOLN:

$$\text{For } \lambda = 4 \text{ we have } A^T A - 4I = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -2 & -1 & 0 \\ -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ So it has two}$$

$$\text{eigenvectors: } v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

For  $\lambda = 2$  we have  $A^T A - 2I = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 0 & -1 & 0 \\ -1 & -1 & 1 & -1 \\ -1 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  so this is the

eigenvector  $v_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$  and, finally, for  $\lambda = 0$  we have

$A^T A - 0I = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  and eigenvector  $v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

As a bonus, we have the diagonalization

$$A^T A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -1 & -2 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & 3 & -1 \\ -1 & 1 & -1 & 1 \\ 0 & -2 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$