Math 13 – Spring '13 – Liberal Arts Mathematics – Chapters 19&20 Test Name____________________

Write your answers to the following questions with thorough explanations written in complete sentences.

1. You may remember having to work problems like, "If Joe can dig a ditch in 3 days, and Sam can dig it in 4, how long will it take the two of them working together?" The answer is related to the harmonic mean of 3 and 4. The formula for the harmonic mean of two numbers *x* and *y* is

$$\frac{2}{\frac{1}{x}+\frac{1}{y}}$$

   a. Calculate the answer for Joe and Sam, which is one-half of the harmonic mean of 3 and 4. Explain why this is the correct answer.
   SOLN: Joes rate of work is $\frac{1}{3}$ of the job per day while Sam's rate is $\frac{1}{4}$ of the job per day. Their combined rate is $\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$. Using the formula, amount done = rate * time we have that $\text{time}=\frac{\text{amount done}}{\text{rate}}=\frac{1}{7/12}=\frac{12}{7}$ days or one week and 5 days.

   b. Show that the harmonic mean of two positive numbers is always less than or equal to the geometric mean. (Thus, in light of Exercise 19 for chapter 19, we have the general conclusion that $H \leq G \leq A$, where *H* stands for the harmonic mean, *G* for the geometric mean, and *A* for the arithmetic mean.) (*Hint:* Suppose that the claim is false. Simplify the fraction that is the harmonic mean, square both sides of the inequality, and proceed as in Exercise 19.)
   SOLN: We wish to show that

$$\frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \sqrt{xy}$$

   Start by showing that the arithmetic mean is greater than the geometric mean, that is

$$\frac{x+y}{2} \geq \sqrt{xy}$$

   This is pretty straightforward to show either algebraically:

$$\frac{x+y}{2} \geq \sqrt{xy} \Leftrightarrow (x+y)^2 \geq 4xy \Leftrightarrow (x-y)^2 \geq 0$$

   Or geometrically (the rectangle that contains the greatest area is a square, say, or the chord of a circle with the greatest length is a diameter.)

   c. Show once more that the harmonic mean of two positive numbers is always less than the geometric mean, but this time do it with less work: let *A* = 1/*x* and *B* = 1/*y*, and discover one connection (equation) between the harmonic mean of *x* and *y* and the arithmetic mean of *A* and *B*, and a second connection between the geometric mean of *x* and *y* and the geometric mean of *A* and *B*. Then use Exercise 19 on *A* and *B*.
   SOLN: Well, this may be a little out of the indicated order, but everything is covered, so, well, everything has to be enough, right? If we let *A* = 1/*x* and *B* = 1/*y*, then

$$\frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \sqrt{xy} \Leftrightarrow \frac{2}{A+B} \leq \frac{1}{\sqrt{AB}} \Leftrightarrow \frac{A+B}{2} \geq \sqrt{AB}$$

   which is what we showed in part (b) above.

   d. What should be the formula for the harmonic mean of three numbers? Of *n* numbers?
   SOLN: The generalization is

$$\frac{2}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} \leq \sqrt{xyz}$$

for three numbers, and

$$\frac{2}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}} \leq \sqrt{x_1 x_2 \cdots x_n}$$

for *n* numbers. This is best understood as a statement that the n-dimensional rectangle with a fixed perimeter maximizes its hypervolume when $x_1 = x_2 = \cdots = x_n$

2. Just as the golden mean arises as the limiting ratio of consecutive terms of the Fibonacci sequence, each of the metallic means arises as the limiting ratio of consecutive terms of generalized Fibonacci sequences. A generalized Fibonacci sequence *G* can be defined by

$$G_1 = 1, G_2 = 1, \text{and } G_{n+1} = pG_n + qG_{n-1},$$

where *p* and *q* are positive integers. The Fibonacci sequence itself is the case *p* = *q* = 1.

a. Try various small values of *p* and *q* and determine which mean they lead to.
SOLN: For $p = q = 1$ this is just the Fibonacci sequence.
Using this program:

```
//G. Hagopian
// This program is designed to prompt the user for initial values p and q and
// then compute the metallic mean for the sequence G_(n+1) = p*G_n + q*P_(n-1)

#include <iostream>
#include <iomanip>
using namespace std;

int main() {
  double p, q, G[100]={0}, toler = 1.e-10;
  cout << "\ninput p and q and we'll compute a limit for the ratio"
       << "\nG(n+1)/G(n) where G(1) = G(2) = 1 and G(n+2) = p*G(n+1) + q*G(n): ";
  while(1)
  {
     cin >> p >> q;
     G[0] =1; G[1] = 1;
     G[2] = p*G[1] + q*G[0];
     int i = 3;
     while(G[i-1]/G[i-2] - G[i-2]/G[i-3] > toler
        || G[i-2]/G[i-3] - G[i-1]/G[i-2] > toler)
     {
           G[i] = p*G[i-1] + q*G[i-2];
           ++i;
     }
     cout << setprecision(10) << "\nThe ratio is " << G[i-1]/G[i-2] << endl;
  }
}
```

The following results were produced:

```
input p and q and we'll compute a limit for the ratio
G(n+1)/G(n) where G(1) = G(2) = 1 and G(n+2) = p*G(n+1) + q*G(n): 1 1
```

b. Divide the equation for G_{n+1} by Assume that G_{n+1}/G_n and G_n/G_{n-1} both tend toward the same number x as n gets large, replace those quantities by x, and simplify the resulting equation. What must be the value of x?

SOLN:

In general, if the ratio has converged then

$$\frac{G_{n+1}}{G_n} = \frac{G_n}{G_{n-1}} = x$$

So dividing $G_{n+1} = pG_n + qG_{n-1}$ through by G_n gives us $x = p + \frac{q}{x} \Leftrightarrow x^2 - px = q$ or, after completing the square,

$$\left(x - \frac{p}{2}\right)^2 = \frac{4q + p^2}{4}$$

So that $x = \frac{p+\sqrt{p^2+4q}}{2}$. Let's try out with some of the empirical values show at right.

If $p = q = 1$ then we get the golden ratio, ϕ.

If $p = 5$ and $q = 2$ then $x = \frac{5+\sqrt{25+8}}{2} \approx 5.372281$

Sure enough!

```
1 1 The ratio is 1.618033989
1 2 The ratio is 2
2 1 The ratio is 2.414213562
1 3 The ratio is 2.302775638
3 1 The ratio is 3.302775638
2 3 The ratio is 3
3 2 The ratio is 3.561552813
4 1 The ratio is 4.236067977
1 4 The ratio is 2.561552813
4 2 The ratio is 4.449489743
2 4 The ratio is 3.236067978
4 3 The ratio is 4.645751311
3 4 The ratio is 4
5 1 The ratio is 5.192582404
1 5 The ratio is 2.791287848
5 2 The ratio is 5.372281323
2 5 The ratio is 3.449489743
5 3 The ratio is 5.541381265
3 5 The ratio is 4.192582404
5 4 The ratio is 5.701562119
4 5 The ratio is 5
6 1 The ratio is 6.16227766
1 6 The ratio is 3
6 2 The ratio is 6.31662479
2 6 The ratio is 3.645751311
6 3 The ratio is 6.464101615
3 6 The ratio is 4.372281323
6 4 The ratio is 6.605551275
4 6 The ratio is 5.16227766
6 5 The ratio is 6.741657387
5 6 The ratio is 6
```

c. What happens to the sequence and to the mean if we allow one or both of p and q to be negative integers?

SOLN: Based on the results shown at right, there appears to be a tendancy for the ratio to diverge. Consider $p = -1, q = 1$. Then the sequence is just Fibonacci with every other term negative: 1,1,0,1,-1,1,-2,3,-5,8,-13,… and the ratio goes to $-\phi$.

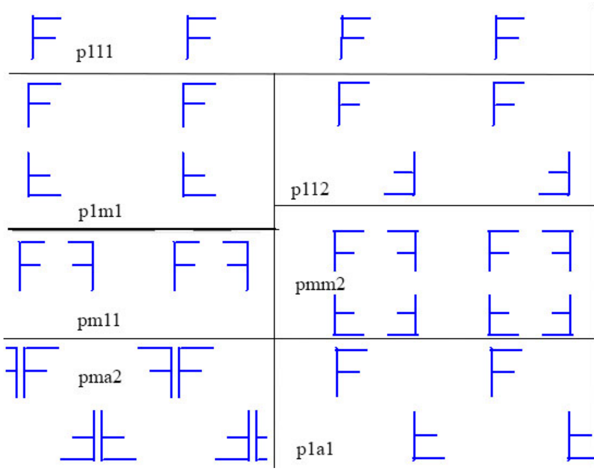
If we set $p = 1, q = -1$ then the sequence

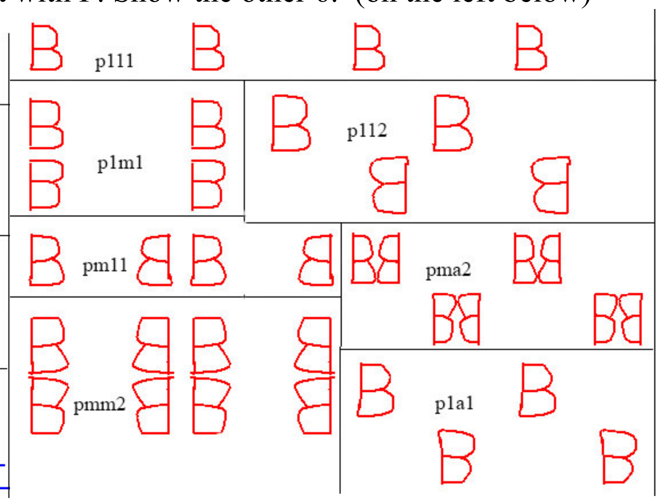
```
-1 1  The ratio is -1.618033989
-1 -1 The ratio is 3.414213562
-2 1  The ratio is -2.414213562
 2 -1 The ratio is 1
-2 -2 The ratio is -1.2676506e+030
-1 -2 The ratio is 7.275390825e+030
-2 -1 The ratio is 121203.0017
-3 2  The ratio is -3.561552813
-3 -2 The ratio is -2
-2 -3 The ratio is 7.940036783e+048
```

Goes like 1,1,0,-1,-1,0,1,1,0,0,… If $p = q = -1$. Then, following the analysis above, $x = \frac{-1+\sqrt{1-4}}{2}$ is not a real number…and the sequence of values is 1, 1, -2, 1, 1, -2 is repeating, but not converging. In the case of $p = -2, q = 1$, $x = \frac{2+\sqrt{4+4}}{2} = 1 + \sqrt{2}$ although apparently negative.

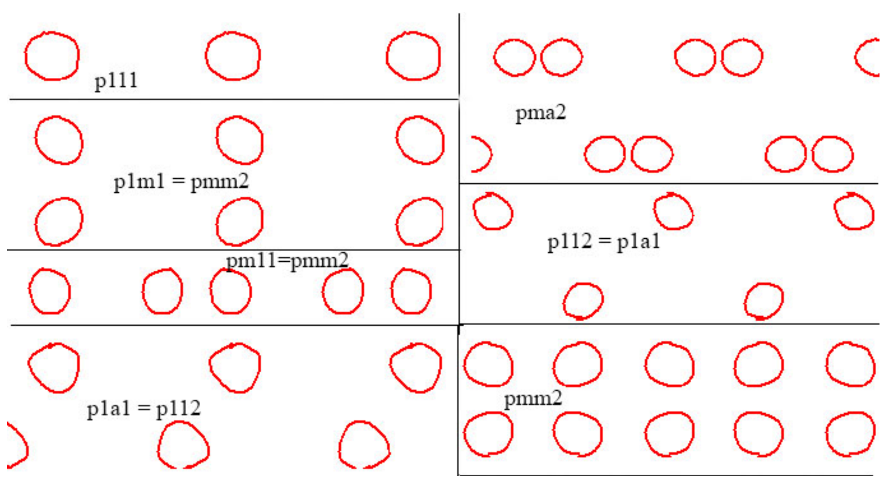
3. Give the notation (such as *d4* or *c5*) for the symmetry patterns of the rosettes in the hubcaps below, disregarding the logos in the centers:
SOLN:
(A) d3, (B) c3, (C) d8, (D) c20, (E) c7, (F) d4, (G) d14, (H) c8, (I) d3, (J) c3

4. .
   a. What is the symmetry group for the following border pattern: ... FFFFFFFFFFFFFFFFFFFFFF...
   SOLN: p111
   b. You can form all 7 border patterns if you start with F. Show the other 6. (on the left below)





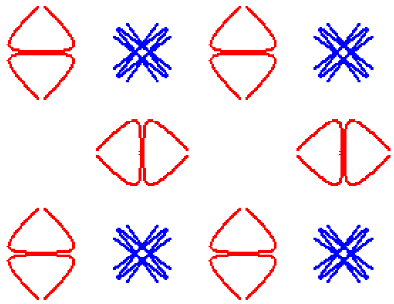
   c. What is the symmetry group for the following border pattern: ... BBBBBBBBBBBBBBBBBBB...
   SOLN: This is a little ambiguous since the B is not entirely symmetric about the horizontal line through its middle. Lets assume it is, then it's a p1m1, otherwise it's just a p111.
   d. You can form all 7 border patterns if you start with B. Show the other 6.
   SOLN: On the right above.
   e. What is the symmetry group for the following border pattern: ... OOOOOOOOOOOOOOOOOO...
   SOLN: This is pmm2



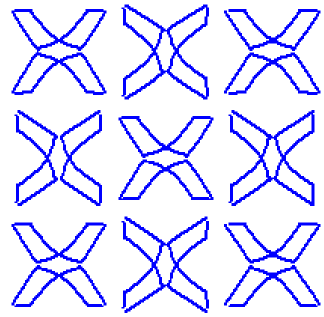
   f. You can form all 7 border patterns if you start with O. Show the other 6.

5. Identify each wallpaper pattern by its symmetry group:

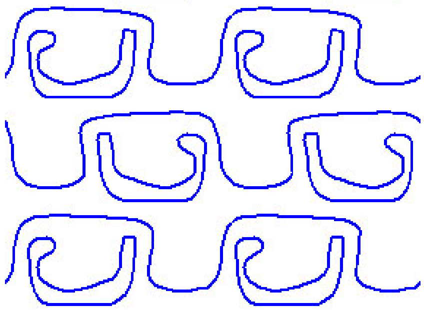
a. SOLN: pmm



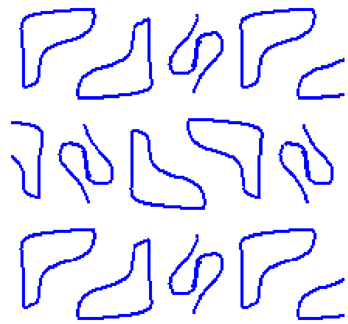
b. SOLN: pmm



c. SOLN: p1

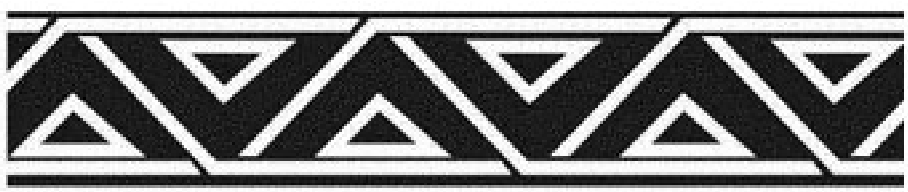


d. SOLN: pgg

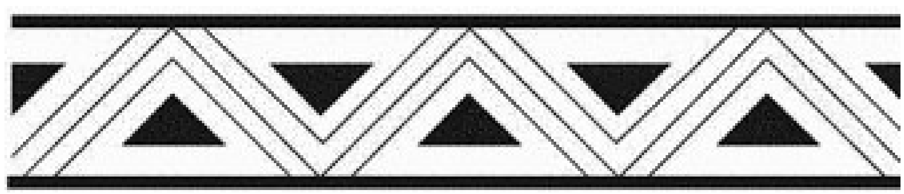


6. What are the elements of the group of symmetries of

a. SOLN: p111

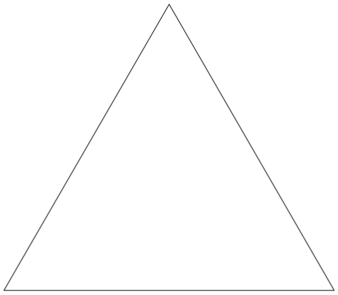


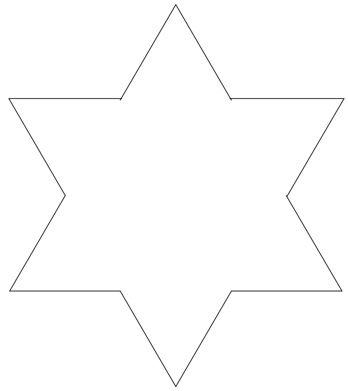
b. SOLN: pma2



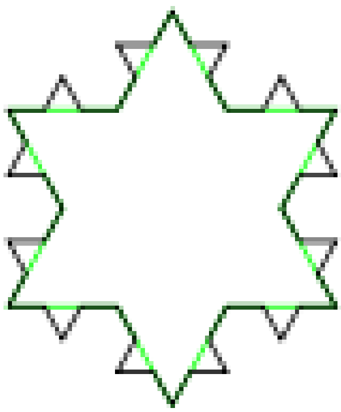
7. Start with a white equilateral triangle. (We will say that this measures 1 X 1 X 1 unit.) Then repeatedly apply the *Iteration Rule:* Divide each side into three equal segments. Replace the middle section with two equal lengths bulging outwards as shown below.

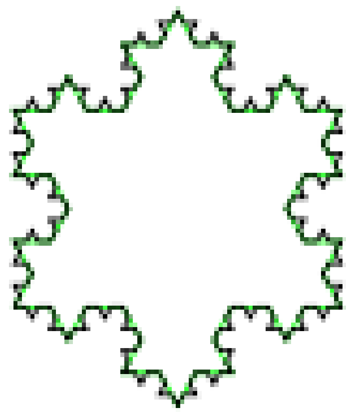
Initial state: First iteration:





a. Draw the 2nd and 3rd Iterations,





b. Complete the table:

Iteration	Number of Sides of Figure	Perimeter	Area contained by the perimeter
0	**3**	**3**	$A_0 = \frac{\sqrt{3}}{2}$
1	$\mathbf{4 \cdot 3}$	$\mathbf{(4 \cdot 3)\frac{1}{3} = 4}$	$A_0 + 3 \cdot \frac{A_0}{9}$
2	$\mathbf{4^2 \cdot 3}$	$\mathbf{4^2 \cdot 3 \cdot \frac{1}{3^2} = \frac{16}{3}}$	$A_0 + 3 \cdot \frac{A_0}{9} + 4 \cdot 3 \cdot \frac{A_0}{81} = A_0\left(1 + \frac{1}{3} + \frac{4}{27}\right)$
3	$\mathbf{4^3 \cdot 3}$	$\mathbf{4^3 \cdot 3 \cdot \frac{1}{3^3} = \frac{64}{9}}$	$A_0 + 3 \cdot \frac{A_0}{9} + 4 \cdot 3 \cdot \frac{A_0}{27} + 4^2 \cdot 3 \cdot \frac{A_0}{729}$

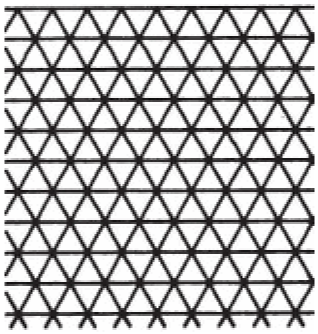
c. Find formulas for the perimeter and the area contained at the *nth* iteration.

SOLN: Each time the perimeter is multiplied by $\frac{4}{3}$ so the formula for the perimeter of the *nth* iteration is $P_n = 3 \cdot \left(\frac{4}{3}\right)^n$. Also, at each iteration, the number of new triangles at the n^{th} iteration is $3 \cdot 4^{n-1}$ each one with an area with is $\frac{1}{9}$ of the previous new triangles' areas. Thus a formula for the total area of the figure is

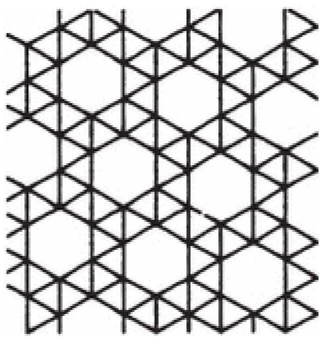
$$A_n = A_0 + 3 \cdot \frac{A_0}{9} + 4 \cdot \frac{A_0}{27} + 4^2 \cdot \frac{A_0}{243} + \cdots + 4^{n-1} \cdot \frac{3A_0}{9^n}$$

8. The usual notation for a vertex type is to denote a regular n-gon by n, separate the sizes of polygons by periods, and list the polygons in clockwise order starting from the smallest number of sides, so that, e.g., 3.3.3.3.3.3 denotes six equilateral triangles meeting at a vertex. The possible vertex types are 3.3.3.3.3.3, 3.3.3.3.6, 3.3.3.4.4, 3.3.4.3.4, 3.3.4.12, 3.4.3.12, 3.3.6.6, 3.6.3.6, 3.4.4.6, 3.4.6.4, 3.12.12, 4.4.4.4, 4.6.12, 4.8.8, 5.5.10, and 6.6.6. Which of these vertex types do not occur in a semi-regular tiling? Explain, drawing picture to make your points clear.

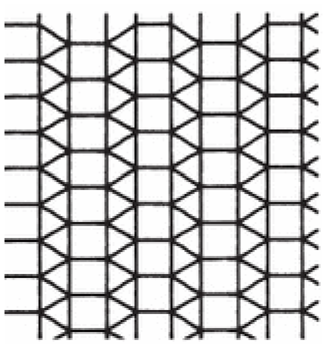
SOLN: For 3.3.3.3.3.3 there are 6 60° angles that come together: $6 \cdot 60 = 360$. That's this pattern:



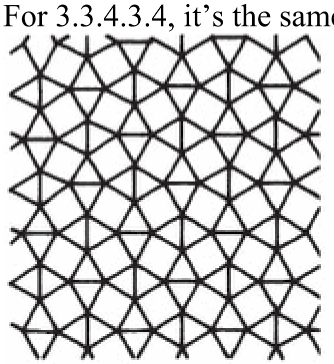
For 3.3.3.3.6 there are 4 60° and one 120° angles: $4 \cdot 60 + 120 = 360$. That's this pattern:



For 3.3.3.4.4 there are 3 60° and two 90° angles: $3 \cdot 60 + 2 \cdot 90 = 360$. That's this pattern:

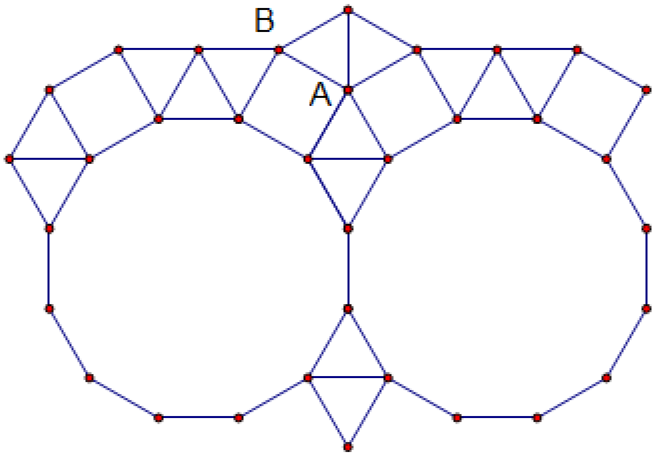


For 3.3.4.3.4, it's the same deal, only in a different order.



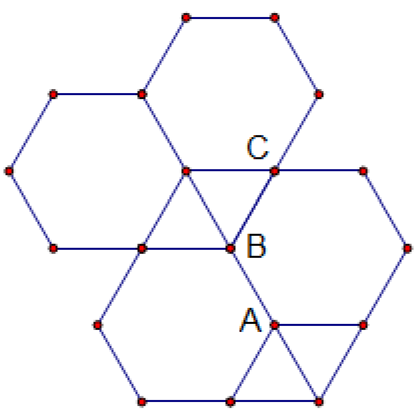
For 3.3.4.12, there are two 60°, a 90° and a 150°: $2 \cdot 60 + 120 + 150 = 360$.

This is where the trouble is. When you try to construct this figure, you run into problems. Consider the construction shown below:

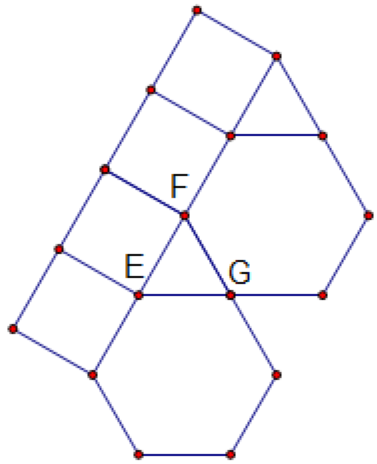


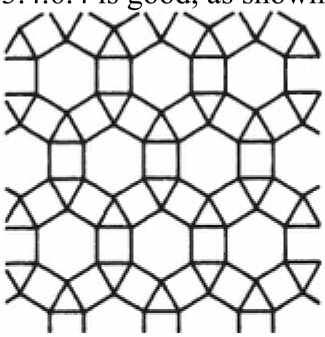
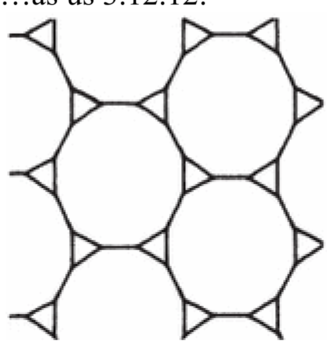
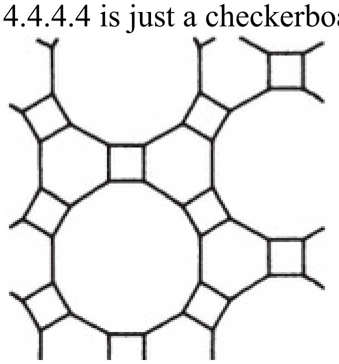
Not only do you need a new vertex type at A (3.3.4.3.4) but you'll have the same problem at B. This does not meet the definition of a semi-regular tiling, to wit: A systematic tiling that uses a mix of regular polygons with different numbers of sides but in which all vertex types are alike—the same polygons in the same order, clockwise or counterclockwise—is called a <u>semiregular tiling</u>.

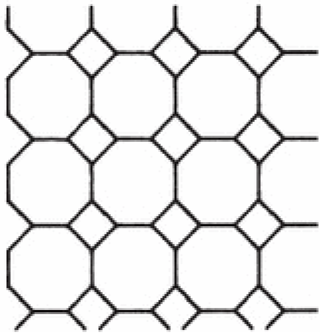
For 3.3.6.6 the sum is $2 \cdot 60 + 2 \cdot 120 = 360$ but again, there are problems in the construction. Here it's a little more obvious, since is we construct these angles around points A and B, then clearly point C has to be of type 3.6.3.6.



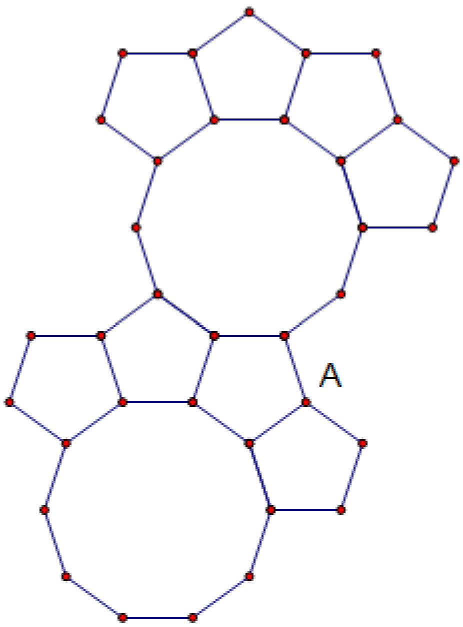
This also explains why 3.6.3.6 doesn't work as a vertex type in a semi-regular tiling.
For 3.4.4.6 we end up with the construction below:



Where G is forced to be a different vertex type.
3.4.6.4 is good, as shown below.

…as us 3.12.12:

4.4.4.4 is just a checkerboard, and 4.6.12 is the semiregular tiling below

The vertex type 4.8.8 adorns many a bathroom floor (below)

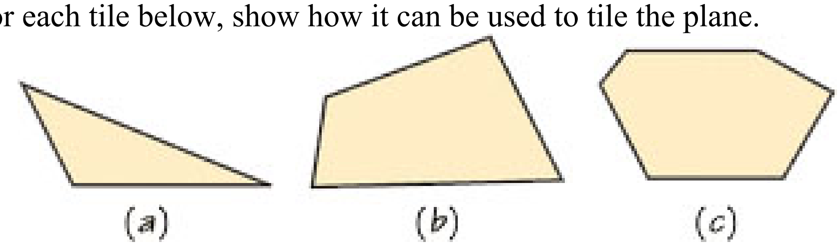


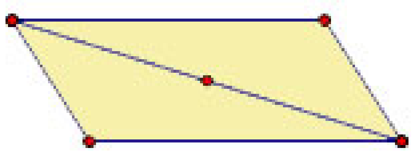
5.5.10 looks promising until you get to vertex A in the diagram below. Not going to work:



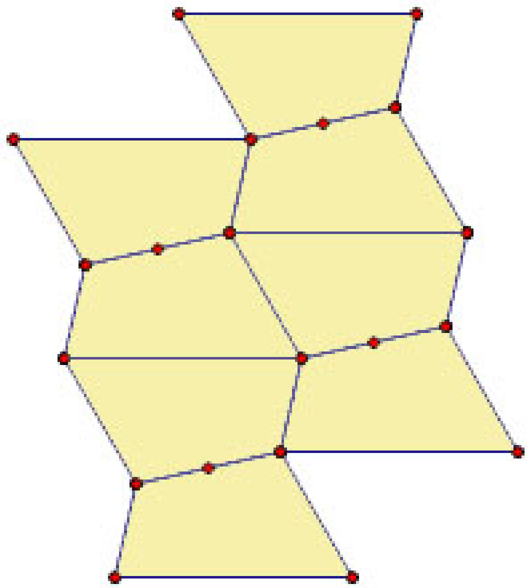
Of course 6.6.6 is the honeycomb. Flash news: I've got one in my olive tree right now!

9. For each tile below, show how it can be used to tile the plane.

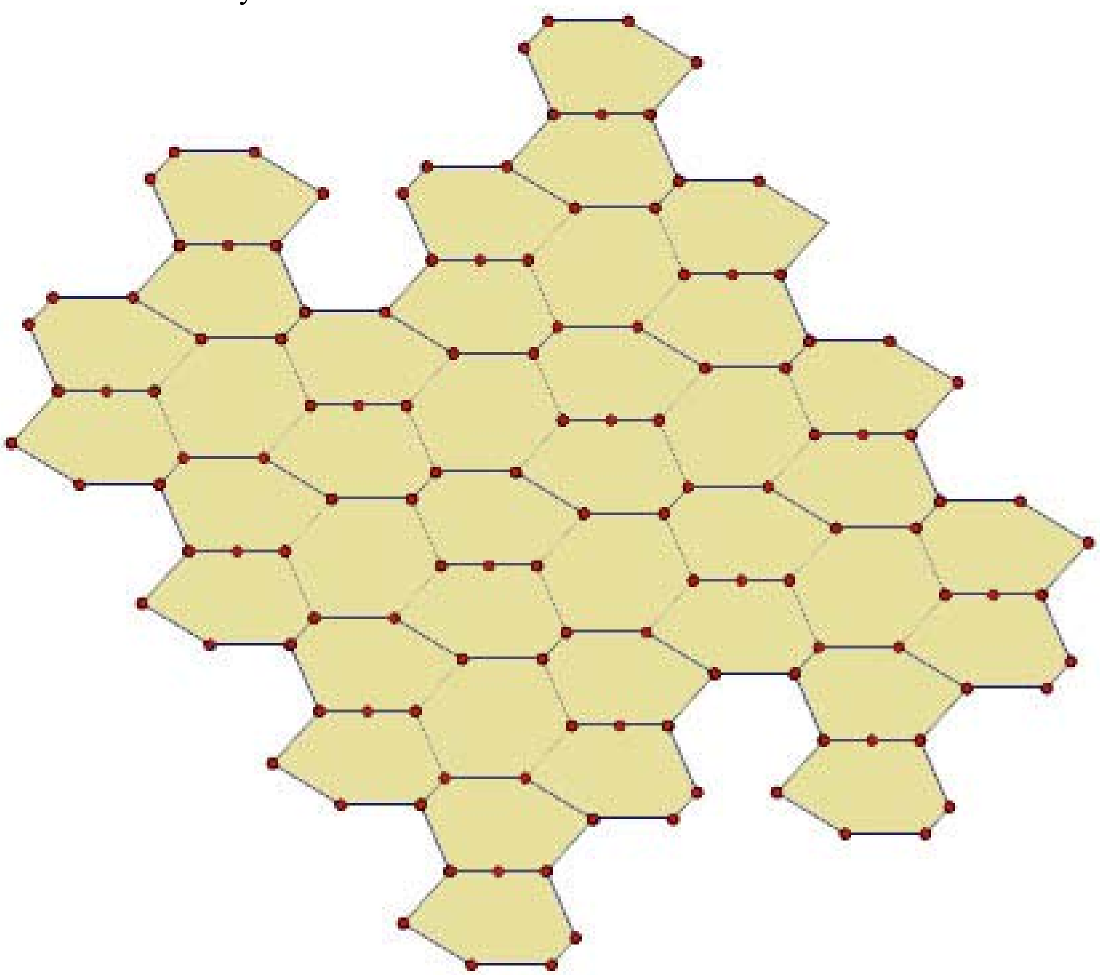


SOLN: (a) pick any side and construct the midpoint on that side. That produces a parallelogram like this  which makes an obvious tessellation.

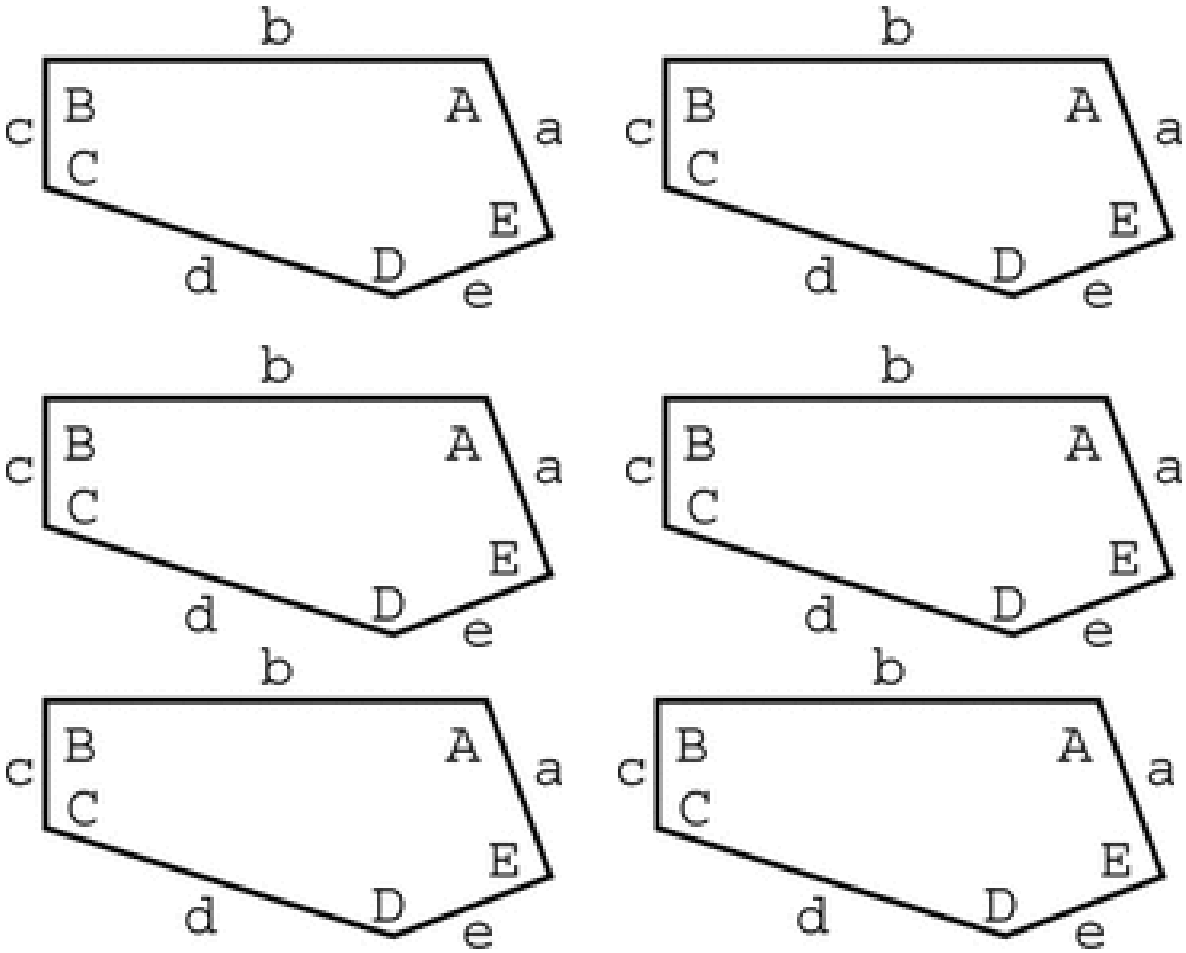
(b) Same basic idea: Construct a midpoint on one side rotate the figure 180° around that point. This then becomes a hexagon with each pair of opposite sides parallel and the figure tessellates nicely:



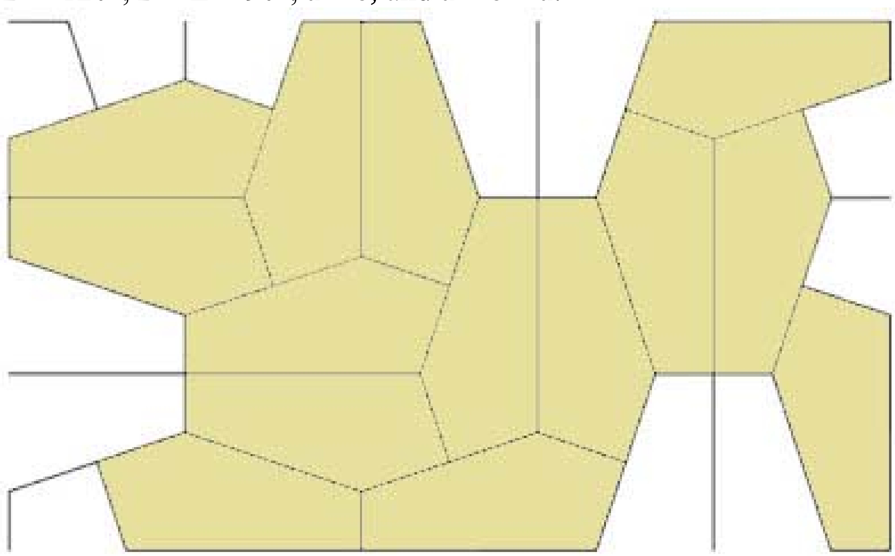
(c) This hinges on the assumption that there are two pairs of opposite sides on the hexagon which are parallel. Pick one of them, construct the midpoint and rotate 180° about that point to get a figue that tessellates by translation:



10. The following is a pentagonal tile of type 13, which was discovered by Marjorie Rice. Show how it can tile the plane. (*Hint:* Carefully trace and cut out a dozen or so copies and try fitting them together.)



The parts of this pentagon satisfy the following relations:
$A = C = D = 120°$, $B = E = 90°$, $a = e$, and $a + e = d$.



SOLN: