Math 13 – Spring '13 – Liberal Arts Mathematics – Chapters 19&20 Test Name_____

Write your answers to the following questions with thorough explanations written in complete sentences.

1. You may remember having to work problems like, "If Joe can dig a ditch in 3 days, and Sam can dig it in 4, how long will it take the two of them working together?" The answer is related to the harmonic mean of 3 and 4. The formula for the harmonic mean of two numbers x and y is

$$\frac{2}{\frac{1}{x}+\frac{1}{y}}$$

   a. Calculate the answer for Joe and Sam, which is one-half of the harmonic mean of 3 and 4. Explain why this is the correct answer.

   b. Show that the harmonic mean of two positive numbers is always less than or equal to the geometric mean. (Thus, in light of Exercise 19 for chapter 19, we have the general conclusion that $H \leq G \leq A$, where H stands for the harmonic mean, G for the geometric mean, and A for the arithmetic mean.) (*Hint:* Suppose that the claim is false. Simplify the fraction that is the harmonic mean, square both sides of the inequality, and proceed as in Exercise 19.)

   c. Show once more that the harmonic mean of two positive numbers is always less than the geometric mean, but this time do it with less work: let $A = 1/x$ and $B = 1/y$, and discover one connection (equation) between the harmonic mean of x and y and the arithmetic mean of A and B, and a second connection between the geometric mean of x and y and the geometric mean of A and B. Then use Exercise 19 on A and B.

   d. What should be the formula for the harmonic mean of three numbers? Of n numbers?

2. Just as the golden mean arises as the limiting ratio of consecutive terms of the Fibonacci sequence, each of the metallic means arises as the limiting ratio of consecutive terms of generalized Fibonacci sequences. A generalized Fibonacci sequence G can be defined by

$$G_1 = 1, G_2 = 1, \text{ and } G_{n+1} = pG_n + qG_{n-1},$$

   where p and q are positive integers. The Fibonacci sequence itself is the case $p = q = 1$.

   a. Try various small values of p and q and determine which mean they lead to.

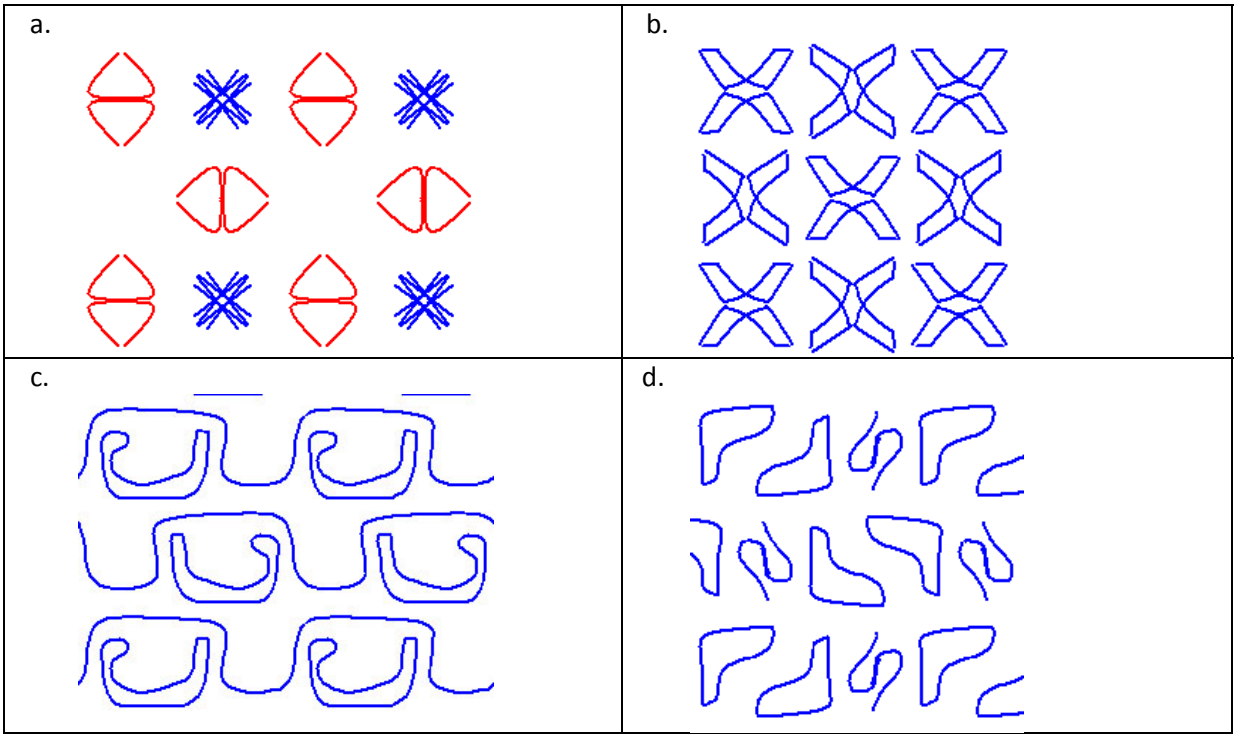
   b. Divide the equation for G_{n+1} by Assume that G_{n+1}/G_n and G_n/G_{n-1} both tend toward the same number x as n gets large, replace those quantities by x, and simplify the resulting equation. What must be the value of x?

   c. What happens to the sequence and to the mean if we allow one or both of p and q to be negative integers?

3. Give the notation (such as *d4* or *c5*) for the symmetry patterns of the rosettes in the hubcaps below, disregarding the logos in the centers.

4. .
   a. What is the symmetry group for the following border pattern: ... FFFFFFFFFFFFFFFFFFFFF...
   b. You can form all 7 border patterns if you start with F. Show the other 6.
   c. What is the symmetry group for the following border pattern: ... BBBBBBBBBBBBBBBBBBB...
   d. You can form all 7 border patterns if you start with B. Show the other 6.
   e. What is the symmetry group for the following border pattern: ... OOOOOOOOOOOOOOOOO ...
   f. You can form all 7 border patterns if you start with O. Show the other 6.

5. Identify each wallpaper pattern by its symmetry group:



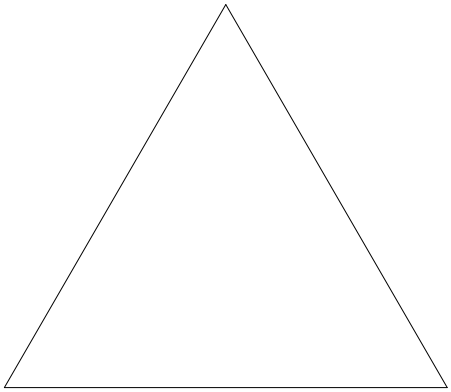
6. What are the elements of the group of symmetries of

   a.

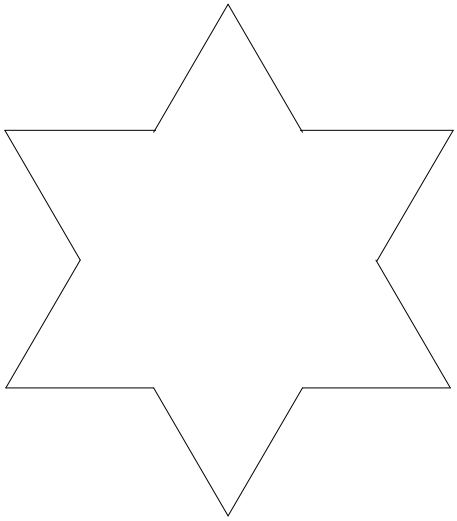
   b.

7. Start with a white equilateral triangle. (We will say that this measures 1 X 1 X 1 unit.) Then repeatedly apply the *Iteration Rule:* Divide each side into three equal segments. Replace the middle section with two equal lengths bulging outwards as shown below.

   Initial state:



   First iteration:



   a. Draw the 2nd and 3rd Iterations,

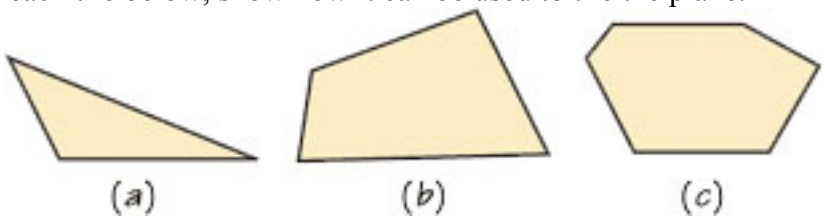
   b. Complete the table:

Iteration	Number of Sides of Figure	Perimeter	Area contained by the perimeter
0			
1			
2			
3			

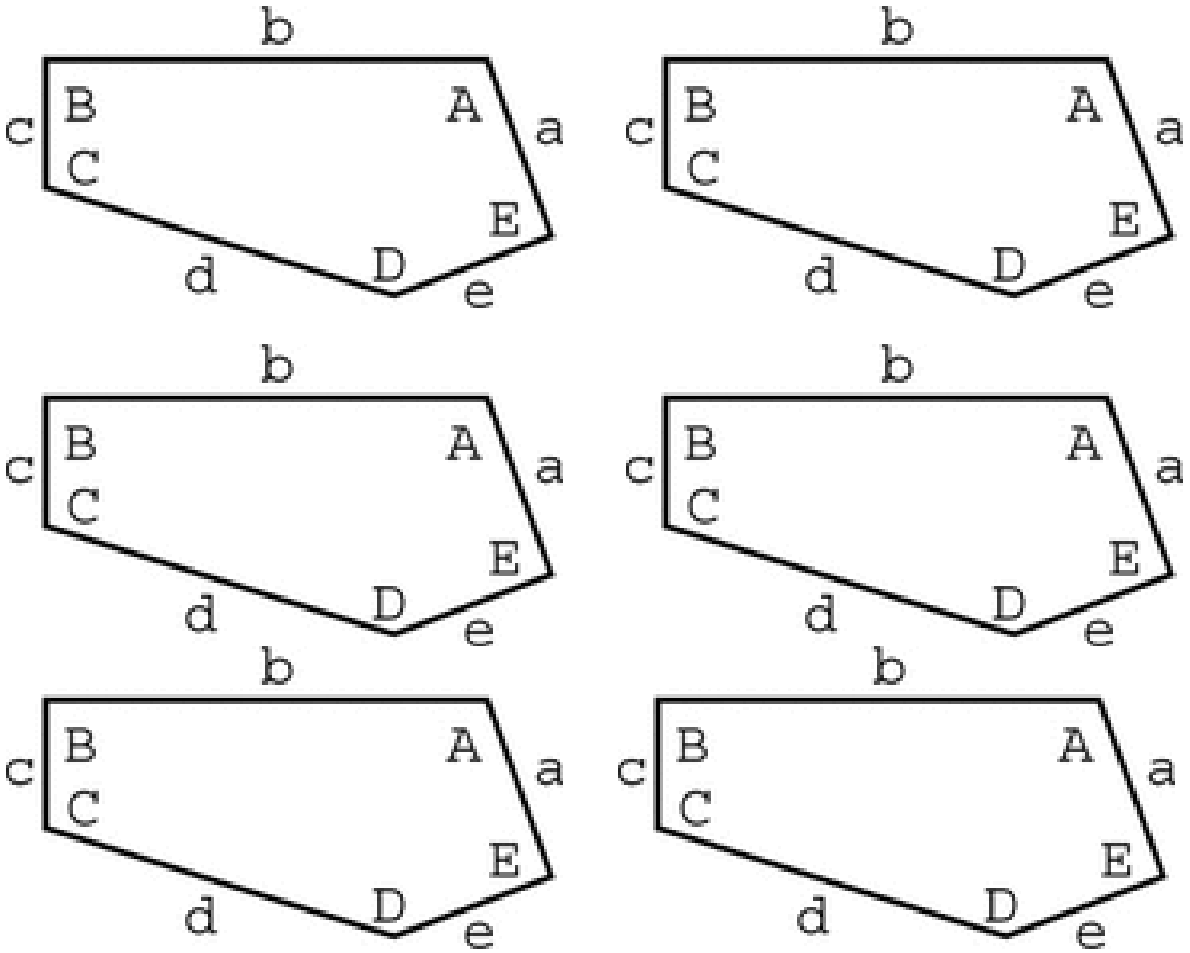
   c. Find formulas for the perimeter and the area contained at the *nth* iteration.

8. The usual notation for a vertex type is to denote a regular *n*-gon by *n,* separate the sizes of polygons by periods, and list the polygons in clockwise order starting from the smallest number of sides, so that, e.g., 3.3.3.3.3.3 denotes six equilateral triangles meeting at a vertex. The possible vertex types are 3.3.3.3.3.3, 3.3.3.3.6, 3.3.3.4.4, 3.3.4.3.4, 3.3.4.12, 3.4.3.12, 3.3.6.6, 3.6.3.6, 3.4.4.6, 3.4.6.4, 3.12.12, 4.4.4.4, 4.6.12, 4.8.8, 5.5.10, and 6.6.6. Which of these vertex types do not occur in a semi-regular tiling? Explain, drawing picture to make your points clear.

9. For each tile below, show how it can be used to tile the plane.



10. The following is a pentagonal tile of type 13, which was discovered by Marjorie Rice. Show how it can tile the plane. (*Hint:* Carefully trace and cut out a dozen or so copies and try fitting them together.)



The parts of this pentagon satisfy the following relations:
$A = C = D = 120°$, $B = E = 90°$, $a = e$, and $a + e = d$.