Math 13 – Spring 13 – Test 2 (Social Choice) Name_____

Write all responses on separate paper. Use complete sentences to articulate your answers.

1. Thirty board members must vote on five candidates: X, Y, Z, U, and V. Their preference rankings are summarized in the table below
2.

	Number of Members		
	12	10	8
First choice	X	Y	Z
Second choice	U	Z	U
Third choice	Y	X	X
Fourth choice	Z	U	V
Fifth choice	V	V	Y

   a. Find the winner using the Borda count.
   b. Find the winner using the Hare system.
   c. Find the winner using sequential pairwise voting with the agenda X, Y, Z, U, V.
   d. Can you set an agenda that will make V the winner?

3. *The Method of Pairwise Comparisons*

   The Method: Every candidate is matched on a one-to-one basis with every other candidate. If candidate X beats candidate Y in one-to-one competition, then X receives one point and Y receives 0 points. If there is a tie, then both X and Y receives 1/2 point each. The candidate with the most points after all of these match-ups is the winner.

   a. What is the outcome of this method as applied to the following distribution of preference ballots? Note that there are 5 candidates so there are 5 choose 2 = 10 one-to-one matches (circle the winner of each): A vs B, A vs C, A vs D, A vs E, B vs C, B vs D, B vs E, C vs D, C vs E and D vs E.

Number of Voters	5	3	5	3	2	4
1st Choice	A	A	C	D	D	B
2nd Choice	B	D	E	C	C	E
3rd Choice	C	B	D	B	B	A
4th Choice	D	C	A	E	A	C
5th Choice	E	E	B	A	E	D

   b. Does this method satisfy the Condorcet criterion? Explain.
   c. Now assume the votes must be recounted. Before this takes place, B, C, and D lose heart and drop out, leaving only A and E. Now what happens?

Number of Voters	10	12
1st Choice	A	E
2nd Choice	E	A

   d. Does this violate the independence of irrelevant alternatives criterion?

4. Consider the following preference table for four voters:

First choice	B	D	A	C
Second choice	C	B	D	D
Third choice	A	C	B	B
Fourth choice	D	A	C	A
Fifth choice	E	E	E	E

   a. If a Borda Count is used, can the first voter manipulate the outcome so that B wins? Explain.
   b. If a Borda Count is used, can the third voter manipulate the outcome so that A wins? Explain.
   c. If a Borda Count is used, can the fourth voter manipulate the outcome so that C wins? Explain.
   d. If a Borda Count is used, can the first voter manipulate the outcome so that B wins? Explain.
   e. If a Borda Count is used, can the second voter manipulate the outcome so that B wins? Explain.

5. Consider the following preference table for three voters:

First choice	A	D	B
Second choice	D	C	A
Third choice	B	B	D
Fourth choice	C	E	C
Fifth choice	E	A	E

   Is there an agenda for which D would win a sequential pairwise election? If so, what is it?

6. A committee has three members with the system [10: 5, 4, 3, 2, 1].
   Which voters are critical in the coalition {A, B, C, E}?

7. Given the weighted voting system [6: 4, 3, 2, 1],
   a. List all the permutations of voters ABCD and circle the pivotal voter in each.
   b. Calculate the Shapely Shubick power index.
   c. List all voting coalitions and indicate the critical voters in each.
   d. Calculate the Banzhaf power index.

8. In the weighted voting system [8: 5, 2, 2, 2, 2, 2],
   a. How many extra votes does the coalition {B, C, E, F} have?
   b. Which voters are critical in the coalition {A, C, D}?
   c. Find the Banzhaf power index for voter B.

Math 13 – Spring 13 – Test 2 (Social Choice) – Solutions.

1. Thirty board members must vote on five candidates: X, Y, Z, U, and V. Their preference rankings are summarized in the table below

	Number of Members		
	12	10	8
First choice	X	Y	Z
Second choice	U	Z	U
Third choice	Y	X	X
Fourth choice	Z	U	V
Fifth choice	V	V	Y

   a. Find the winner using the Borda count.
   SOLN: X gets 4*12+2*10+2*8=84, Y gets 2*12+4*10=64, Z gets 1*12+3*10+4*8=74, U gets 3*12+1*10+3*8=70 and V gets 1*8=8 so X is the winner.
   b. Find the winner using the Hare system.
   SOLN: First U and V are eliminated, since they are tied for last with 0 first place votes. Then Z is eliminated (only 8 first place votes) and finally X wins 20 to 10.
   c. Find the winner using sequential pairwise voting with the agenda X, Y, Z, U, V.
   SOLN: In X vs Y, Y is eliminated. It's then X vs. Z, which Z wins by 18 to 12. In fact, Z holds on to that 18 in every subsequent match so that Z is the winner.

2. *The Method of Pairwise Comparisons*
   The Method: Every candidate is matched on a one-to-one basis with every other candidate. If candidate X beats candidate Y in one-to-one competition, then X receives one point and Y receives 0 points. If there is a tie, then both X and Y receives 1/2 point each. The candidate with the most points after all of these match-ups is the winner.
   a. What is the outcome of this method as applied to the following distribution of preference ballots?
   Note that there are 5 candidates so there are 5 choose 2 = 10 one-to-one matches (circle the winner of each): A vs B, A vs C, A vs D, A vs E, B vs C, B vs D, B vs E, C vs D, C vs E and D vs E.

Number of Voters	5	3	5	3	2	4
1st Choice	A	A	C	D	D	B
2nd Choice	B	D	E	C	C	E
3rd Choice	C	B	D	B	B	A
4th Choice	D	C	A	E	A	C
5th Choice	E	E	B	A	E	D

	A	B	C	D	E
A		1	1	1	0
B	0		1	0	1
C	0	0		1	1
D	0	1	0		1
E	1	0	0	0	

   SOLN:
   The table to the right above tabulates 1 if the row beats the column, otherwise 0. So we add the rows to see A gets 3, C gets 2, D gets 2 and E gets 1. So A is the winner.
   b. Does this method satisfy the Condorcet criterion? Explain.
   SOLN: A voting system satisfies the Condorcet criterion if it chooses the Condorcet winner when one exists. Clearly, if there is a Condorcet winner, then it will also be a Method of Pairwise Comparisons winner.
   c. Now assume the votes must be recounted. Before this takes place, B, C, and D lose heart and drop out, leaving only A and E. Now what happens?
   SOLN: Clearly, E is now the winner.

Number of Voters	10	12
1st Choice	A	E
2nd Choice	E	A

   d. Does this violate the independence of irrelevant alternatives criterion?
   SOLN: Yes, candidate E has moved from a nonwinner status to winner status without any voter changing their ranking.

3. Consider the following preference table for four voters:

First choice	B	D	A	C
Second choice	C	B	D	D
Third choice	A	C	B	B
Fourth choice	D	A	C	A
Fifth choice	E	E	E	E

   a. If a Borda Count is used, can the first voter manipulate the outcome so that B wins? Explain.
   SOLN: Yes, the current count has A with 8, B with 11, C with 10 and D with 11. By moving D down a notch, the first voter can decrease D's count to 10, making B the winner.

   b. If a Borda Count is used, can the third voter manipulate the outcome so that A wins? Explain.
   SOLN: No. B and D both have 11 to A's 8. Thus, while the third voter can decrease D's vote to 8, and B's count to 10, that doesn't make A a winner.

   c. If a Borda Count is used, can the fourth voter manipulate the outcome so that C wins? Explain.
   SOLN: Yes. If the fourth voter ranks the candidates CEADB, then the count comes out as
   A with 9, B with 9, C with 10 and D with 9…and E also ran.

   d. If a Borda Count is used, can the second voter manipulate the outcome so that B wins? Explain.
   SOLN: Yes, but only by making their preferred candidate, D, the loser. This is not a successful manipulation.

4. Consider the following preference table for three voters:

First choice	A	D	B
Second choice	D	C	A
Third choice	B	B	D
Fourth choice	C	E	C
Fifth choice	E	A	E

   Is there an agenda for which D would win a sequential pairwise election? If so, what is it?
   SOLN: Yes. BACED does it. B beats A then C and then E, but D beats B. Also EBACD and a number of others. The only way to make D lose is to put D before A without B intervening.

5. A committee has three members with the system [10: 5, 4, 3, 2, 1].
   Which voters are critical in the coalition {A, B, C, E}?
   SOLN: A and B. The coalition has 13 so there are 3 extra votes. Both A and B's weight exceed 3.

6. Given the weighted voting system [6: 4, 3, 2, 1],

a. List all the permutations of voters ABCD and ~~circle~~ underline the pivotal voter in each.
SOLN: A<u>B</u>CD, A<u>B</u>DC, A<u>C</u>BD, A<u>C</u>DB, AD<u>B</u>C, AD<u>C</u>B, B<u>A</u>CD, B<u>A</u>DC, BC<u>A</u>D, BC<u>D</u>A, BD<u>A</u>C, BD<u>C</u>A, C<u>A</u>BD, C<u>A</u>DB, CB<u>A</u>D, CB<u>D</u>A, CD<u>A</u>B, CD<u>B</u>A, DA<u>B</u>C, DA<u>C</u>B, DB<u>A</u>C, DB<u>C</u>A, DC<u>A</u>B, DC<u>B</u>A

b. Calculate the Shapely Shubick power index.
SOLN: $\left(\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{12}\right)$

c. List all voting coalitions and indicate the critical voters in each.
SOLN:

d. Calculate the Banzhaf power index.
SOLN: (10,6,6,2)

binary	coalition	Critical
1111	ABCD	
1110	ABC	A
1101	ABD	AB
1011	ACD	AC
0111	BCD	BCD
1100	AB	AB
1010	AC	AC
1001	AD	BC
0110	BC	AD
0101	BD	AC
0011	CD	AB
1000	A	BC
0100	B	A
0010	C	A
0001	D	
0000		

7. In the weighted voting system [8: 5, 2, 2, 2, 2, 2],
   a. How many extra votes does the coalition {B, C, E, F} have?
   SOLN: None. Zip. Nada. Zilch.
   b. Which voters are critical in the coalition {A, C, D}?
   SOLN: Each one. A, C and D are each one a critical voter.
   c. Find the Banzhaf power index for voter B.
   SOLN: There are 6 voters, so there are $2^6 = 64$ different coalitions. Rather than list them all, we do all little deduction. B is critical to a coalition of the type {B,2,2,2} and there are 4 ways to choose the other three twos from among the remaining four twos, once B is chosen. Also, B is critical to a coalition of the form {B,5,2} and there are 4 ways to choose which 2 of the remaining four twos will be chose once B is chosen. So B's BPI value is 8.