Math 13 – Liberal Arts Math – HW7 – Chapter 11

1. Give an example of a weighted voting system that has a dummy voter but no dictator that is not [6:5,3,1].

2. Explain why the weighted voting system [13: 10, 6, 5, 3, 2] is not a legitimate weighted voting system.
3. Give an example of a weighted voting system that has a blocking coalition that would not be a winning coalition if all its members voted YES.
4. Given the weighted voting system [30: 20, 17, 10, 5], list all winning coalitions.
5. Given the weighted voting system [51: 45, 43, 7, 5],
   a. list all blocking coalitions.
   b. list all minimal winning coalitions
6. Given the weighted voting system [30: 20, 17, 10, 5],
   a. list all minimal winning coalitions.
   b. list all blocking coalitions.
7. A weighted voting system has four voters, A, B, C, and D. List all possible coalitions of these voters. How many such coalitions are there?
8. In a weighted voting system, is a voter with veto power the same as a dictator? Why or why not?
9. A weighted voting system has 12 members. How many distinct coalitions are there in which exactly seven members vote YES?
10. Given the weighted voting system [5: 3, 2, 1, 1, 1], find which voters of the coalition {A, C, D, E} are critical?
11. Given the weighted voting system [8: 5, 4, 3], find the Banzhaf power index for each voter.
12. Given the weighted voting system [14: 10, 6, 5, 3], find the Banzhaf power index for each voter.
13. Given the weighted voting system [7: 4, 1, 1, 1, 1, 1], find the Banzhaf power index for each voter.
14. Give an example of a weighted voting system that is equivalent to the [15: 8, 7, 6].
15. What is the difference between a "critical" voter in a coalition and a "pivotal" voter in a permutation?
16. Calculate the Shapely-Shubik power index for the weighted voting system
   a. [30: 20, 17, 10, 5].
   b. [8: 6, 1, 1, 1, 1, 1].
17. There are five distinct three-member voting systems. Give an example of each.
18. Given the weighted voting system [4: 1, 2, 3],
   a. list all winning coalitions.
   b. list all blocking coalitions.
19. Given the weighted voting system [16: 3, 9, 4, 5, 10], calculate the Banzhaf power index for each voter.
20. Given the weighted voting system [14: 8, 2, 5, 7, 4], calculate the Shapley-Shubik power index for each voter.

Math 13 – Liberal Arts Math – HW7 Solutions – Chapter 11

1. Give an example of a weighted voting system that has a dummy voter but no dictator that is not [6:5,3,1].
   SOLN: One solution is [9: 6, 5, 2], or [100: 98, 2, 1]
2. Explain why the weighted voting system [13: 10, 6, 5, 3, 2] is not a legitimate weighted voting system.
   SOLN: The system given is not a legitimate weighted voting system because the quota is exactly half of the total vote weight. Two different complementary coalitions exist with vote weight total of 13, (A, D) and (B, C, E).
3. Give an example of a weighted voting system that has a blocking coalition that would not be a winning coalition if all its members voted YES.
   SOLN. One solution is: In the [14: 10, 6, 5, 3, 2], the coalition (A, D) is a blocking coalition because (B, C, E) has only 13 votes. (A, D) would not be a winning coalition by voting "yes" because (A, D) has only 13 votes.
4. Given the weighted voting system [30: 20, 17, 10, 5], list all winning coalitions.
   SOLN: (A, D) (A, C) (B, C, D) (A, B, C) (A, B, D) (A, C, D) (A, B, C, D).
5. Given the weighted voting system [51: 45, 43, 7, 5],
   a. list all blocking coalitions.
      SOLN: (A, B) (A, C) (A, B, C) (A, B, D) (A, C, D) (A, D) (B, C) (B, C, D) (A, B, C, D)
   b. list all minimal winning coalitions
      SOLN: (A, B) (A, C) (B, C, D)
6. Given the weighted voting system [30: 20, 17, 10, 5],
   a. list all minimal winning coalitions.
      SOLN: (A, B) (A, C) (B, C, D)
   b. list all blocking coalitions.
      SOLN: (A, B) (A, C) (A, B, C) (A, B, D) (A, C, D) (B, C, D) (A, B, C, D) (A, D) (B, C)
7. A weighted voting system has four voters, A, B, C, and D. List all possible coalitions of these voters. How many such coalitions are there?
   SOLN: There are 16 coalitions possible from four voters:
   Ø, (A) (B) (C) (D) (A, B) (A, C) (A, D) (B, C) (B, D) (C, D) (A, B, C) (A, B, D) (A, C, D) (B, C, D) (A, B, C, D)
8. In a weighted voting system, is a voter with veto power the same as a dictator? Why or why not?
   SOLN: NO: A voter with veto power has enough votes to block any measure, but not necessarily enough to pass any issue. A dictator has enough votes to pass any issue on his or her own
9. A weighted voting system has 12 members. How many distinct coalitions are there in which exactly seven members vote YES?

$$12C_7 = \frac{12!}{7!\,5!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2} = 11 \cdot 9 \cdot 8 = 792$$

10. Given the weighted voting system [5: 3, 2, 1, 1, 1], find which voters of the coalition {A, C, D, E} are critical?

SOLN: Since the coalition {A, C, D, E} has one extra vote, the only critical member is voter A with weight 3.

11. Given the weighted voting system [8: 5, 4, 3], find the Banzhaf power index for each voter.
SOLN: There are 6 permutations of the voters:
ABC, ACB, BAC, BCA, CAB, CBA. A is pivotal in BAC, BCA, CAB and CBA while B is pivotal in ABC and C is pivotal in ACB, so the SSPI is $\left\{\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right\}$. But we're asked for the Banzhaf Power Index…For that we make a listing of all possible voting coalitions, of which there are $2^3 = 8$.
As the table below indicates, the BPI is {6,2,2}.

A	B	C	Critical
1	1	1	A
1	1	0	AB
1	0	1	AC
0	1	1	A
1	0	0	BC
0	1	0	A
0	0	1	A
0	0	0	

12. Given the weighted voting system [14: 10, 6, 5, 3], find the Banzhaf power index for each voter.
SOLN: Now there are 24 = 16 coalitions (with the critical voters as tabulated below) so the BPI is {10,6,6,2}.

A	B	C	D	Critical	A	B	C	D	Critical
1	1	1	1		0	1	1	0	A D
1	1	1	0	A	0	1	0	1	A C
1	1	0	1	AB	0	0	1	1	AB
1	0	1	1	A C	0	0	0	1	
0	1	1	1	BCD	0	0	1	0	A
1	1	0	0	AB	0	1	0	0	A
1	0	1	0	A C	1	0	0	0	BC
1	0	0	1	BC	0	0	0	0	

13. Given the weighted voting system [7: 4, 1, 1, 1, 1, 1], find the Banzhaf power index for each voter.
SOLN: Now there are 26 = 64 different coalitions, but rather than list them all, we not that there are $\binom{5}{3} = 10$ ways a group of weight-1 voters can be a critical part of a blocking coalition and $\binom{5}{3} = 10$ ways a pair of weight-1 voters can be a critical part of a winning coalition, so each of the weight-1 voters has BPI of 20. Also, voter A is critical in the winning coalitions 3, 4, or 5 weight-1 voters, of which there are 10+5+1=16 and critical in the blocking coalitions involving 0, 1, or 2 weight-1 voters, of which there are 1+5+10=16. Thus the BPI is {32,20,20,20,20,20}.
14. Give an example of a weighted voting system that is equivalent to the [15: 8, 7, 6].
SOLN: There are many good solutions here. One solution is: [32: 20, 15, 10].

15. What is the difference between a "critical" voter in a coalition and a "pivotal" voter in a permutation?

    SOLN: A critical voter in a winning or blocking coalition is any voter who has sufficient weight so that the coalition would no longer be winning or blocking were this voter to switch their vote. The order of voters in the coalition does not matter. There can be more than one critical voter in a coalition. A pivotal voter is the first voter who joins a coalition and gives that coalition enough votes to win. Each permutation has exactly one pivotal voter.

16. Calculate the Shapely-Shubik power index for the weighted voting system

    a. [30: 20, 17, 10, 5].

       SOLN: There are 4! = 24 permutations, so it's not too hard to list them all (see below.)

       Tallying up we see the SSPI is $\left\{\frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}\right\}$

A**B**CD	AD**B**C	BC**A**D	C**A**BD	CD**A**B	DB**A**C
AB**D**C	AD**C**B	BC**D**A	CA**D**B	CDB**A**	DB**C**A
A**C**BD	BA**C**D	BDA**C**	CBA**D**	DA**B**C	DCA**B**
A**C**DB	BA**D**C	BDC**A**	CBD**A**	DA**C**B	DCB**A**

    b. [8: 6, 1, 1, 1, 1, 1].

       SOLN: Here there will be 6! = 720 permutations, but we can categorize them as (1) permutations in which 6 goes first or second, in each case there are 4! = 24 ways to rearrange the weight-1 voters around a particular pivotal weight-1 voter, so each weight-1 voter is pivotal in 48 different permutations. Since this is true for each of the 5 weight-1 voters, there are 240 permutations in which A is not pivotal. A is thus pivotal in 720 – 240 = 480 permutations. Thus the SSPI is $\left\{\frac{480}{720}, \frac{48}{720}, \frac{48}{720}, \frac{48}{720}, \frac{48}{720}, \frac{48}{720}\right\} = \left\{\frac{2}{3}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}\right\}$

17. There are five distinct three-member voting systems. Give an example of each.

    Answers may vary. One example of each of the five distinct voting systems is:

    [3: 3, 1, 1] dictator
    [4: 2, 2, 1] two with veto power
    [2: 1, 1, 1] each voter is equal, majority rules
    [3: 2, 1, 1] one with veto power
    [3: 1, 1, 1] unanimous vote required

18. Given the weighted voting system [4: 1, 2, 3],

    a. list all winning coalitions.

       SOLN: {A, C}, {B, C}, {A, B, C}

    b. list all blocking coalitions.

       SOLN: {C}, {A, B}, {A, C}, {B, C}, {A, B, C}

19. Given the weighted voting system [16: 3, 9, 4, 5, 10], calculate the Banzhaf power index for each voter.

    SOLN: (4, 8, 4, 4, 8) There are 25 = 32 coalitions (itemized with critical voters below.)

    So the BPI is {8,16,8,8,16}.

coalition	CV		CV		CV		CV
11111		10110	B E	11000	CDE	00101	AB D
11110	B	01110	BCD	10100	B E	00011	ABC

11101		11001	B E	10010	B E	10000	
11011		10101	A C E	10001	BCD	01000	E
10111	E	01101	B E	01100	A DE	00100	
01111		10011	A DE	01010	A C E	00010	
11100	ABC	01011	B E	01001	B E	00001	B
11010	AB D	00111	CDE	00110	B E	00000	

20. Given the weighted voting system [14: 8, 2, 5, 7, 4], calculate the Shapley-Shubik power index for each voter. SOLN: Here there are 5! = 120 permutations. Yikes. Here it goes!
$\left\{\frac{42}{120}, \frac{12}{120}, \frac{22}{120}, \frac{32}{120}, \frac{12}{120}\right\} = \left\{\frac{7}{20}, \frac{1}{10}, \frac{11}{60}, \frac{4}{15}, \frac{1}{10}\right\}$

AB**C**DE	A**D**CBE	BA**C**DE	BD**C**AE	CB**A**DE	CD**A**BE	DB**C**AE	D**A**CBE	EBC**D**A	ED**C**BA
AB**C**ED	A**D**CEB	BA**C**ED	BD**C**EA	CB**A**ED	CD**A**EB	DB**C**EA	D**A**CEB	EBC**A**D	ED**C**AB
AB**D**CE	A**D**BCE	BA**D**CE	BD**A**CE	CB**D**AE	CD**B**AE	DB**A**CE	D**A**BCE	EBD**C**A	EDB**C**A
AB**D**EC	A**D**BEC	BA**D**EC	BD**A**EC	CB**D**EA	CD**B**EA	DB**A**EC	D**A**BEC	EBD**A**C	EDB**A**C
AB**E**DC	A**D**EBC	BA**E**DC	BDE**A**C	CBE**D**A	CD**E**BA	DBE**A**C	D**A**EBC	EB**A**DC	ED**A**BC
AB**E**CD	A**D**ECB	BA**E**CD	BDE**C**A	CBE**A**D	CD**E**AB	DBE**C**A	D**A**ECB	EB**A**CD	ED**A**CB
AC**B**DE	AE**C**DB	BC**A**DE	BEC**D**A	CA**B**DE	CE**A**DB	DC**B**AE	DE**C**AB	ECB**D**A	EA**C**DB
AC**B**ED	AE**C**BD	BC**A**ED	BEC**A**D	CA**B**ED	CE**A**BD	DC**B**EA	DE**C**BA	ECB**A**D	EA**C**BD
AC**D**BE	AE**D**CB	BC**D**AE	BED**C**A	CA**D**BE	CE**D**AB	DC**A**BE	DE**A**CB	EC**D**BA	EA**D**CB
AC**D**EB	AE**D**BC	BC**D**EA	BED**A**C	CA**D**EB	CE**D**BA	DC**A**EB	DE**A**BC	EC**D**AB	EA**D**BC
AC**E**DB	AE**B**DC	BCE**D**A	BE**A**DC	CA**E**DB	CEB**D**A	DC**E**AB	DEB**A**C	EC**A**DB	EA**B**DC
AC**E**BD	AE**B**CD	BCE**A**D	BE**A**CD	CA**E**BD	CEB**A**D	DC**E**BA	DEB**C**A	EC**A**BD	EA**B**CD
02244	02280	30252	70410	32052	52032	52302	90300	70140	32520

That was awful!