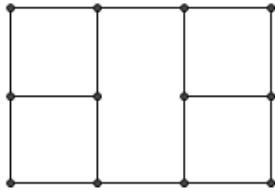


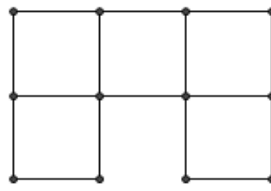
Show your work for credit. Use complete sentences to explain your answers.  
Write all responses on separate paper.

- 1 For each graph below, either prove that it has an Euler circuit or duplicate a minimal number of edges to Eulerize it.

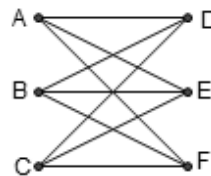
a.



b.

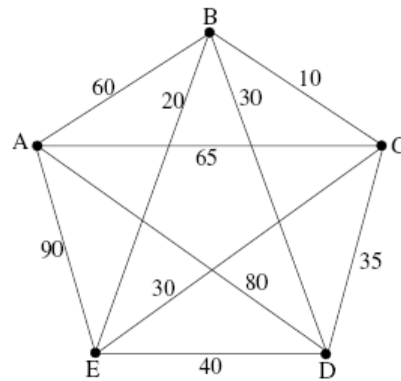


2. The graph shown at right is bipartite. How many different Hamiltonian circuits does it have, assuming the path starts at A? Remember to justify your answer.

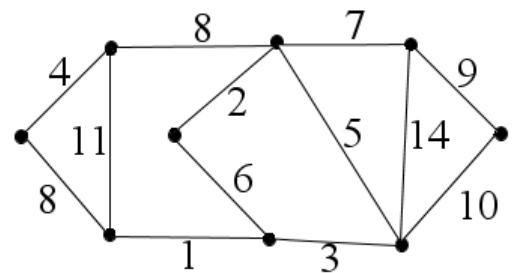


3. For the weighted edge graph shown at right, find the Hamiltonian circuit obtained by using

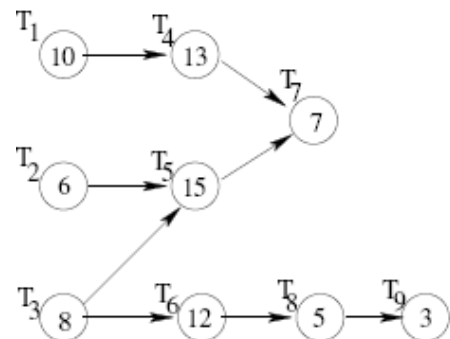
- The greedy algorithm, starting at A.
- The sorted edges algorithm.



4. Apply Kruskal's algorithm to the graph at right to find a minimal spanning tree. Recall Kruskal's algorithm: add links in order of cheapest cost so that no circuits form and every vertex belongs to some link added.



5. Consider the order requirement digraph shown at right.
- What is (are) the critical path(s)?
  - What is the minimal amount of time to complete the task with an unlimited number of processors?
  - What is the minimal time to complete the task with 1 processor?
  - What is the minimal time to complete the task with 2 processors?



6. Suppose the following package sizes are to be packed into bins of size 20 using the worst fit heuristic (put the item into the bin that has the most room available). What will the filled bins look like (reading from left to right?)

4 2 8 2 3 3 2 7 4 3 4 5 7 2 3 4 6 5 7 2 9 1 8 2 3 9 5 1 9 9

7. The table at right shows pairs of actors (x) that can't work together without coming to blows. Draw a graph that would be used to schedule plays for a summer stock. How many plays are required? Hint: Find the chromatic number of the graph.

	A	B	C	D	E	F	G
A		X		X		X	X
B	X		X	X		X	
C		X			X	X	
D	X	X			X	X	
E			X	X			
F	X	X	X	X			X
G	X					X	

8. Use the preference ballots at right to calculate the winner of an election by
- Plurality voting
  - The Hare system
  - The Borda count
  - Pairwise sequential voting with agenda ACB

	number of voters		
rank	6	2	3
first	A	B	C
second	B	C	D
third	C	D	B
fourth	D	A	A

9. In which of the votes for #8 above does the winner agree with the Condorcet winner?

10. Prior to an election a small poll is taken, whose results are shown in the table at right. If in a second poll, the five voters in the first column were to reverse their ranking of B and C, how would the outcome of a Borda count change and what does this have to do with the principle of irrelevance of independent alternatives?

number of voters:	5	3	3
first choice	A	B	C
second choice	C	A	B
third choice	B	C	A

11. A corporation has four shareholders and a total of 100 shares. The quota for passing a measure is the votes of shareholders owning 51 or more shares. The number of shares owned are as follows  
 A: 48 shares    B: 23 shares    C: 22 shares    D: 7 shares  
 Thus the voting system is [51: 48, 23, 22, 7]
- Make a table of winning coalitions, etc. to compute the Banzhaf power index for this voting system.
  - List all permutations of the voters and circle the pivotal voter in each to find the Shapely Shubik power index.
  - How many shares can A sell to B without changing any of the winning coalitions?

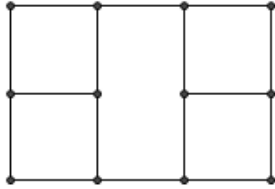
12. Use the Huffman compression algorithm to construct a Huffman tree for the phrase, "one teen ate at a tee." Start by making a frequency count of the letters. Show the process, step by step. Finally, write the code words for the letters E, T, A, O, and N.

13. Using the Vigenère cipher with the key word GLUE, a message is encrypted as ZSCRMDZERWUTGCN. Decrypt the message.  
 Start by finding the additive inverses modulo 26 of G, L, U and E. (Hint: the additive inverse of 6 is 20).  
 A B C D E F G H I J K L M N O P Q R S T U V W X Y Z  
 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

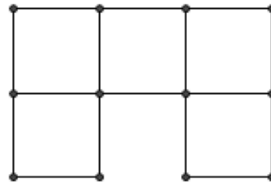
# Math 13 — Final Exam Solutions — Spring 2013

1 For each graph below, either prove that it has an Euler circuit or duplicate a minimal number of edges to Eulerize it.

a.

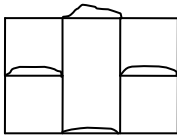


b.

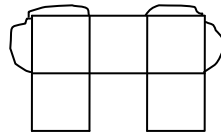


SOLN:

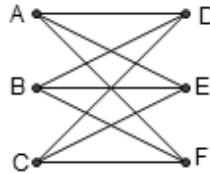
a.



b.



2. The graph shown at right is bipartite. How many different Hamiltonian circuits does it have, assuming the path starts at A? Remember to justify your answer.



SOLN: By the fundamental principle of counting, there are  $3 \cdot 2 \cdot 2 = 12$  different Hamiltonian circuits.

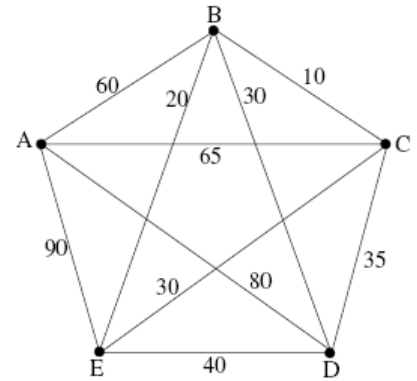
3. For the weighted edge graph shown at right, find the Hamiltonian circuit obtained by using

a. The greedy algorithm, starting at A.

SOLN: ABCEDA

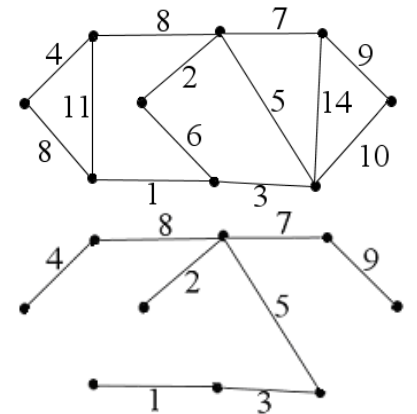
b. The sorted edges algorithm.

SOLN: ADCBEA



4. Apply Kruskal's algorithm to the graph at right to find a minimal spanning tree. Recall Kruskal's algorithm: add links in order of cheapest cost so that no circuits form and every vertex belongs to some link added.

SOLN: Kruskal's algorithm leads to the spanning tree at right:



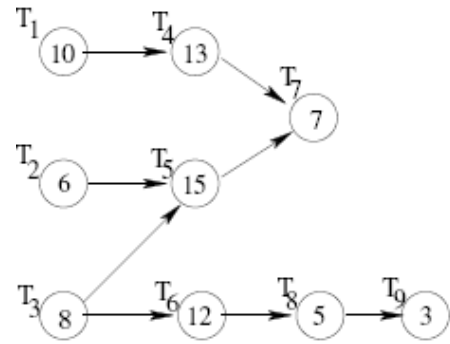
5. Consider the order requirement digraph shown at right.

a. What is (are) the critical path(s)?  
 SOLN:  $T_1T_4T_7$  &  $T_3T_5T_7$  both have maximal length 30

b. What is the minimal amount of time to complete the task with an unlimited number of processors?  
 SOLN: 30

c. What is the minimal time to complete the task with 1 processor?  
 SOLN: 79

d. What is the minimal time to complete the task with 2 processors?  
 SOLN: You can do it in 40:  
 P1:  $T_1-T_4-T_6-T_8$   
 P2:  $T_3-T_2-T_5-T_7-T_9$



6. Suppose the following package sizes are to be packed into bins of size 20 using the worst fit heuristic (put the item into the bin that has the most room available). What will the filled bins look like (reading from left to right?)

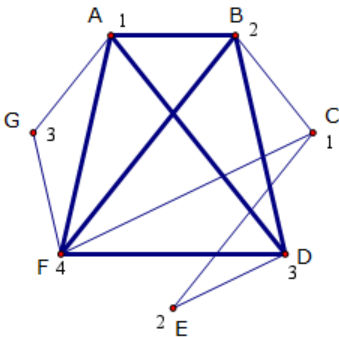
4 2 8 2 3 3 2 7 4 3 4 5 7 2 3 4 6 5 7 2 9 1 8 2 3 9 5 1 9 9

**SOLN: The worst fit bins are 19 19 19 18 18 14 14 18**

7. The table at right shows pairs of actors (x) that can't work together without coming to blows. Draw a graph that would be used to schedule plays for a summer stock. How many plays are required? Hint: Find the chromatic number of the graph.

	A	B	C	D	E	F	G
A		X		X		X	X
B	X		X	X		X	
C		X			X	X	
D	X	X			X	X	
E			X	X			
F	X	X	X	X			X
G	X					X	

SOLN:  $K_4$  (thick lines) is a subgraph so chromatic number is 4:



8. Use the preference ballots at right to calculate the winner of an election by
- Plurality voting SOLN: A wins
  - The Hare system SOLN: A wins
  - The Borda count SOLN: B wins
  - Pairwise sequential voting with agenda ACBD SOLN: A wins

	number of voters		
rank	6	2	3
first	A	B	C
second	B	C	D
third	C	D	B
fourth	D	A	A

9. In which of the votes for #8 above does the winner agree with the Condorcet winner?  
SOLN: (a), (b) and (d)

10. Prior to an election a small poll is taken, whose results are shown in the table at right. If in a second poll, the five voters in the first column were to reverse their ranking of B and C, how would the outcome of a Borda count change and what does this have to do with the principle of irrelevance of independent alternatives?

number of voters:	5	3	3
first choice	A	B	C
second choice	C	A	B
third choice	B	C	A

SOLN: By switching B and C, A is no longer the winner. This shows the Borda count violates the IIA condition since B moved ahead of A without anyone changing their relative ranking of A and B.

11. A corporation has four shareholders and a total of 100 shares. The quota for passing a measure is the votes of shareholders owning 51 or more shares. The number of shares owned are as follows

A: 48 shares    B: 23 shares    C: 22 shares    D: 7 shares

Thus the voting system is [51: 48, 23, 22, 7]

Make a table of winning coalitions, etc. to compute the Banzhaf power index for this voting system.

SOLN: BPI is [12, 4, 4, 4]

a.

Winning Coalition	weight	Extra votes	A	B	C	D
A B C	93	42	1	0	0	0
A B D	78	27	1	0	0	0
A C D	77	26	1	0	0	0
A B	71	20	1	1	0	0
A C	70	19	1	0	1	0
A D	55	4	1	0	0	1
B C D	52	1	0	1	1	1

- b. List all permutations of the voters and circle the pivotal voter in each to find the Shapely Shubik power index.

SOLN:

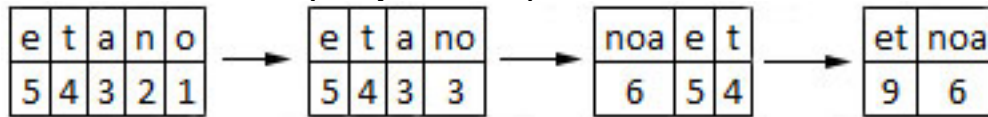
ABCD    BACD    CABD    DABC  
 ABDC    BADC    CADB    DACB  
 ACBD    BCAD    CBAD    DBAC  
 ACDB    BCDA    CBDA    DBCA  
 ADBC    BDAC    CDAB    DCAB  
 ADCB    BDCA    CDBA    DCBA

The SSPI is [1/2, 1/6, 1/6, 1/6]

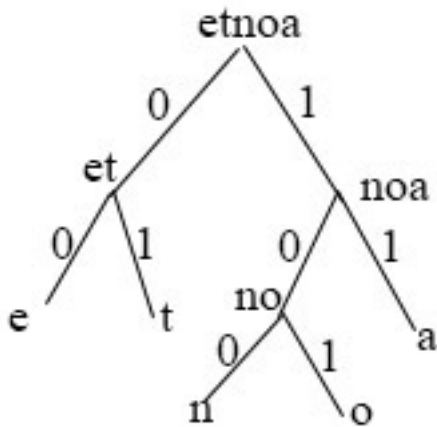
- c. How many shares can A sell to B without changing any of the winning coalitions?

SOLN: If A sells 4 to B, none of the coalitions change. If A sells 5, then AD is no longer a winning coalition.

12. Use the Huffman compression algorithm to construct a Huffman tree for the phrase, "one teen ate at a tee." Start by making a frequency count of the letters. Show the process, step by step. Finally, write the code words for the letters E, T, A, O, and N.
- SOLN: Start with a frequency table compression:



Then build the tree:



So the code words are E = 00, T = 01, N = 100, O = 101 and A = 11.

13. Using the Vigenère cipher with the key word GLUE, a message is encrypted as ZSCRMDZERWUTGCN. Decrypt the message.

Start by finding the additive inverses modulo 26 of G, L, U and E. (Hint: the additive inverse of 6 is 20).

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

SOLN:

Z	S	C	R	M	D	Z	E	R	W	U	T	G	C	N
25	18	2	17	12	3	25	4	17	22	20	19	6	2	13
20	15	6	22	20	15	6	22	20	15	6	22	20	15	6
19	7	8	13	32	18	5	26	11	11	26	41	26	17	19
19	7	8	13	6	18	5	0	11	11	0	15	0	17	19
T	H	I	N	G	S	F	A	L	L	A	P	A	R	T