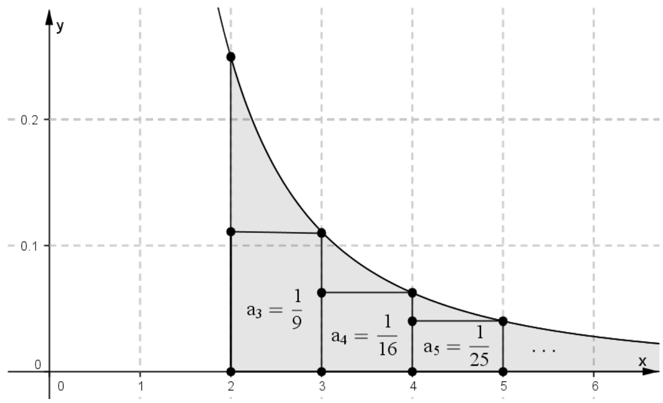
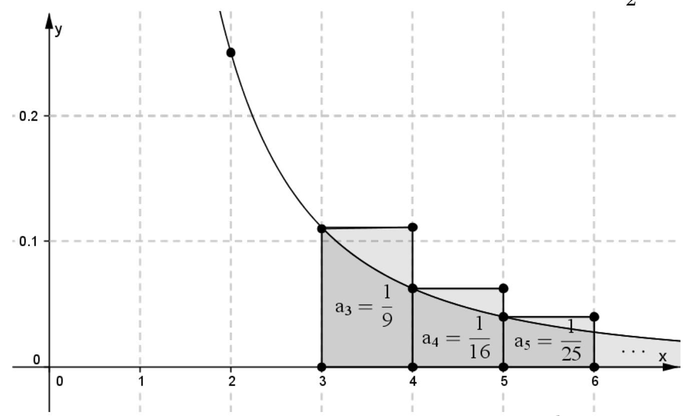
1. (14 points) Find the value of c if $\sum\limits_{n=0}^{\infty} e^{-cn} = 2$.

Solution: $\sum\limits_{n=0}^{\infty} (e^{-c})^n = 2 \Leftrightarrow \dfrac{1}{1-e^{-c}} = 2 \Leftrightarrow c = \ln 2$

2. (18 points) Consider the series $S = \sum\limits_{n=3}^{\infty} \dfrac{1}{n^2}$.

(a) Sketch a diagram (you can use the starter plot below, or make your own) to justify the inequality, $S < \int\limits_2^{\infty} \dfrac{dx}{x^2}$





Solution: As the diagram on the left above shows, the area represented by the integral $\int\limits_2^{\infty} \dfrac{dx}{x^2}$ entirely contains the area of the boxes with base = 1 and heights, $a_3 = \frac{1}{9}, a_4 = \frac{1}{16}, a_5 = \frac{1}{25}, \ldots, a_n = \frac{1}{n^2}, \ldots$ and so the value of the integral is greater than the value of the series.

(b) Make another diagram (or use the other above) to show that $\int\limits_3^{\infty} \dfrac{dx}{x^2} < S$

Solution: The same boxes, shifted one unit to the right, are seen in the second diagram fitting entirely above the area of the integral $\int\limits_3^{\infty} \dfrac{dx}{x^2}$, showing the inequality.

(c) We know that $\zeta(2) = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6} \approx 1.64493$. Use this to determine whether $\int\limits_3^{\infty} \dfrac{dx}{x^2}$ or $\int\limits_2^{\infty} \dfrac{dx}{x^2}$ is closer to S.

Solution: Evidently, $S = \sum\limits_{n=3}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6} - 1 - \dfrac{1}{4} = \dfrac{2\pi^2 - 15}{12} \approx 0.39493$ is closer to $\int\limits_3^{\infty} \dfrac{dx}{x^2} = \dfrac{1}{3}$ than it is to $\int\limits_2^{\infty} \dfrac{dx}{x^2} = \dfrac{1}{2}$. This is also evident from the concavity of f.

3. (16 points) (a) Use the limit comparison test to prove that $\sum\limits_{n=1}^{\infty} \tan\left(\dfrac{1}{n}\right)$ is divergent. *Hint: harmonic series.*

Solution: Since the harmonic series is divergent and
$\lim_{n\to\infty} \dfrac{\tan(1/n)}{1/n} = \lim_{x\to 0} \dfrac{\tan x}{x} = \lim_{x\to 0} \sec^2 x = 1 > 0$ the series $\sum\limits_{n=1}^{\infty} \tan\left(\dfrac{1}{n}\right)$ is also divergent.

(b) Prove that $\sum\limits_{n=0}^{\infty} (-1)^n \tan\left(\dfrac{1}{n+1}\right)$ is convergent. How would you choose N so that $R_N = S - S_N < 0.001$?

Solution: This is an alternating series and nth term is going to zero, so it's convergent. The difference $s - S_N < |a_{N+1}| = \tan\left(\frac{1}{N+2}\right) \approx \frac{1}{N}$ So you'd choose $N = 1000$ for the desired accuracy. To be sure, in Mathematica,

In[22]:= N[Sum[(-1)^n*Tan[1/(n+1)],{n,0,1000}]] Out[22]= 1.21407
whereas,
In[17]:= N[Sum[(-1)^n*Tan[1/(n+1)],{n,0,10000000}]] Out[17]= 1.21357
Not surprisingly, the error is more like half the first neglected term.

4. (24 points) $f(x) = \cos(\pi x)$.

   (a) Write the Maclaurin series for $f(x)$.

   Solution: $\cos(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!}$

   (b) Use the ratio test to find the radius of convergence for the series.

   Solution: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(\pi x)^{2n+2}}{(\pi x)^{2n}} \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} \frac{(\pi x)^2}{(2n+2)(2n+1)} = 0$ for all x, so the radius of convergence is ∞.

   (c) Find the approximating Taylor polynomial of degree 4 for $g(x) = e^x \cos(\pi x)$.

   Hint: multiply two Taylor series.

   Solution: All the terms we want (and then some) are in the expansion of the product

   $$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\right)\left(1 - \frac{\pi^2 x^2}{2} + \frac{\pi^4 x^4}{24}\right) \approx 1 + x - \frac{\pi^2 - 1}{2}x^2 - \frac{3\pi^2 - 1}{6}x^3 + \frac{\pi^4 - 6\pi^2 + 1}{24}x^4$$

5. (12 points) Use Maclaurin series to justify that $e^{\imath x} = \cos(x) + \imath \sin(x)$. Recall that $\imath^2 = -1$.

Solution: $e^{\imath x} = \sum_{n=0}^{\infty} \frac{(\imath x)^n}{n!}$. Group the even and odd terms to get

$$e^{\imath x} = \sum_{n=0}^{\infty} \frac{(\imath x)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(\imath x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\imath(-1)^n x^{2n+1}}{(2n+1)!} = \cos x + \imath \sin x$$

6. (16 points) Recall that the binomial series has

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

   (a) Find the cubic Taylor polynomial approximating $f(x) = \frac{1}{(1+x)^3}$.

   Solution:

$$(1+x)^{-3} = \sum_{n=0}^{\infty} \binom{-3}{n} x^n = 1 - 3x + \frac{-3(-4)}{2!}x^2 + \frac{-3(-4)(-5)}{3!}x^3 + \cdots \approx 1 - 3x + 6x^2 - 10x^3$$

   (b) Consider $\sqrt[3]{1729} = \sqrt[3]{1728 + 1} = 12\sqrt[3]{1 + 1/1728}$ and use the binomial series for $(1+x)^{1/3}$ to approximate this to the nearest millionth. (You can leave your expression in fractional form.)

   Solution: $(1+x)^{1/3} = 1 + \frac{x}{3} + \frac{1/3(1/3-1)}{2}x^2 + \cdots$, so $12\sqrt[3]{1 + 1/1728} = 12 + 4\frac{1}{1728} - \frac{1}{9}\left(\frac{1}{1728}\right)^2 + \cdots$

   $12 + \frac{1}{432} - \frac{1}{26873856} + \cdots \approx 12 + \frac{1}{432}$ is accurate to the nearest millionth since the series will be alternating from this point and the first neglected term is significantly less than a millionth. Thus $\sqrt[3]{1729} \approx 12.0023\overline{148}$. Using a calculator, we get the approximation $\sqrt[3]{1729} \approx 12.002314368427684396585596422711$ is within a millionth of our two-term approximation.