Math 1B **Name (Print):** ____________________

Chapter 7 Test

Write all responses on separate paper. Show your work for credit. Do not use a calculator.

1. (20 points) Evaluate the integral. Make all the substitutions and infinitesimals explicit.

   (a) $\int_0^{\ln 2} xe^{-\pi x}\,dx$. **Solution:**

$u = x$	$dv = e^{-\pi x}dx$
$du = dx$	$v = -\frac{1}{\pi}e^{-\pi x}$

   yields $-\frac{x}{\pi}e^{-\pi x}\Big|_0^{\ln 2} + \frac{1}{\pi}\int_0^{\ln 2} e^{-\pi x}\,dx = -\frac{\ln 2}{\pi 2^\pi} - \frac{1}{\pi^2}e^{-\pi x}\Big|_0^{\ln 2} = -\frac{\ln 2}{\pi 2^\pi} - \frac{1}{\pi^2 2^\pi} + \frac{1}{\pi^2} =$

   $$= \frac{1}{\pi}\left[\frac{1}{\pi} - \frac{1}{2^\pi}\left(\frac{1}{\pi} + \ln 2\right)\right] \approx 0.06484$$

   (b) $\int_1^e t^3 \ln t\,dt$. **Solution:**

$u = \ln t$	$dv = t^3\,dt$
$du = \frac{dt}{t}$	$v = \frac{1}{4}t^4$

   Thus, $\int_1^e t^3\ln(t)\,dt = \frac{t^4}{4}\ln(t)\big|_1^e - \frac{1}{4}\int_1^e t^3\,dt = \frac{e^4}{4} - \frac{t^4}{16}\Big|_1^e =$

   $\frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16} = \frac{3e^4+1}{16} \approx 10.2997$

2. (30 points) Consider the integral function,

   $$I_n(t) = \int_0^t \sin^n(x)\,dx = \int_0^t \sin^{n-1}(x)\sin(x)\,dx$$

   (a) Use integration by parts to show that

   $$I_n(t) = -\frac{1}{n}\cos(t)\sin^{n-1}(t) + \frac{n-1}{n}I_{n-2}(t)$$

   Make an table of u, dv, du, and v. Look for the integral recurring after applying a trig identity.

   Solution:

$u = \sin^{n-1}(x)$	$dv = \sin(x)\,dx$
$du = (n-1)\sin^{n-2}(x)\cos(x)\,dx$	$v = -\cos(x)$

   so that

   $$I_n(t) = -\sin^{n-1}(x)\cos(x)|_0^t + (n-1)\int_0^t \sin^{n-2}(x)\cos^2(x)\,dx$$

   $$= -\sin^{n-1}(t)\cos(t) + (n-1)\int_0^t \sin^{n-2}(x)(1-\sin^2(x))\,dx$$

   $$= -\sin^{n-1}(t)\cos(t) + (n-1)\int_0^t \sin^{n-2}(x)\,dx - (n-1)I_n(t)$$

   $\Leftrightarrow (n-1)I_n(t) + I_n(t) = -\sin^{n-1}(t)\cos(t) + (n-1)I_{n-2}(t)$

   The result follows by combining like terms on the left and dividing through by n.

   (b) Use the reduction formula from (a) to evaluate $\int_0^{\pi/2} \sin^5(x)\,dx$

   Solution: $\int_0^{\pi/2} \sin^5(x)\,dx = \frac{4}{5}\frac{2}{3}\int_0^{\pi/2} \sin(x)\,dx = -\frac{8}{15}\cos(x)\Big|_0^{\pi/2} = \frac{8}{15}$

3. (20 points) Evaluate each integral:

(a)

$$\int_4^5 \frac{x}{\sqrt{x^2-8x+17}}\,dx$$

Solution: $\int_4^5 \frac{x}{\sqrt{x^2-8x+17}}\,dx = \int_4^5 \frac{x}{\sqrt{(x-4)^2+1}}\,dx$ Substitute $u = x-4$ $du = dx$ so that $\int_0^1 \frac{u+4}{\sqrt{u^2+1}}\,du$

Now let $u = \tan\theta$, $du = \sec^2\theta\,d\theta$ so the integral becomes $\int_0^{\pi/4} \frac{4+\tan\theta}{\sec\theta}\sec^2\theta\,d\theta = \int_0^{\pi/4} 4\sec\theta + \sec\theta\tan\theta\,d\theta$

$= 4\ln|\sec\theta+\tan\theta| + \sec\theta\Big|_0^{\pi/4} = 4\ln(\sqrt{2}+1)+\sqrt{2}-1$

(b)

$$\int_2^3 \frac{3x^2+2x}{(x-1)(x^2+2x+2)}\,dx$$

Solution: Start by finding the partial fractions expansion: $\frac{3x^2+2x}{(x-1)(x^2+2x+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+2} \Leftrightarrow$ $A(x^2+2x+2)+(Bx+C)(x-1) = 3x^2+2x$ After expanding and equating coefficients, this leads to the system

$$\boxed{\begin{aligned} A+B&=3\\ 2A-B+C&=2\\ 2A-C&=0 \end{aligned}}$$

However, since this must true for all x, we can plug in $x=1$ to get $5A=5$, so $A=1$, $B=2$ and $C=2$, whence $\int_2^3 \frac{3x^2+2x}{(x-1)(x^2+2x+2)}\,dx = \int_2^3 \frac{1}{x-1} + \frac{2x+2}{x^2+2x+2}\,dx = \ln(x-1)+\ln(x^2+2x+2)\Big|_2^3 =$ $\ln(2)+\ln(17)-\ln(10)$

In retrospect, it would have been easier to multiply out the denominator to get

$\int_2^3 \frac{3x^2+2x}{x^3+x^2-2}\,dx = \ln(x^3+x^2-2)\Big|_2^3 = \ln(34)-\ln(10) = \ln(3.4)$

4. (20 points) Evaluate the improper integral.

(a)

$$\int_0^\infty e^{-x}\sin x\,dx$$

Solution: $\boxed{\begin{matrix} u=e^{-x} & dv=\sin x dx \\ du=-e^{-x}dx & v=-\cos x\end{matrix}} \Rightarrow I = \lim_{b\to\infty} -\cos(b)e^{-b}+1-\int_0^\infty e^{-x}\cos x\,dx$

$\boxed{\begin{matrix} u=e^{-x} & dv=\cos x\,dx \\ du=-e^{-x}dx & v=\sin x\end{matrix}} \Leftrightarrow I = 1-\lim_{b\to\infty} -\sin(b)e^{-b} - I \Leftrightarrow 2I = 1$ so $I = \frac{1}{2}$

(b)

$$\int_{-\infty}^\infty \frac{e^x}{e^{2x}+4}\,dx$$

Solution: Let $2u = e^x$, $2du = e^x dx$ As $x\to\infty$, $u\to\infty$ and as $x\to-\infty, u\to 0$. Thus $\int_{-\infty}^\infty \frac{e^x}{e^{2x}+4}\,dx =$ $\int_0^\infty \frac{2du}{4u^2+4} = \lim_{b\to\infty}\frac{1}{2}\arctan(b) = \frac{\pi}{4}$

5. (10 points) Use comparison to determine whether or not the integral is convergent.

$$\int_1^\infty \frac{\arctan x}{x^4 + x^2}\,dx$$

Solution: $\int_1^\infty \frac{\arctan x}{x^4 + x^2}\,dx =< \int_1^\infty \frac{2}{x^2+1}\,dx = \lim_{b\to\infty} 2\arctan(b) - \frac{\pi}{2} = \frac{\pi}{2}$ So the integral is convergent.