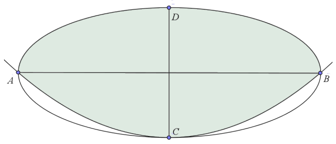
Math 1B Chapter 6 Test Solutions

1. (30 points) Consider the area inside an ellipse with major axis $AB = 4$ and minor axis $CD = 2$, and also above the parabola passing through A, B, and C as shown below .



(a) Introduce a coordinate system and write equations for the ellipse and the parabola relative to your coordinate system. Recall that the standard equation for an ellipse centered at the origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
Solution: Taking the center of the ellipse as the origin and the usual horizontal/vertical x/y coordinate system, the standard form for the equation of the ellipse is $\frac{x^2}{4} + y^2 = 1$. The parabola then has x-intercepts $(\pm 2, 0)$ with a vertex at $(0, -1)$ so its equation can be written as $y = \frac{1}{4}(x^2 - 4)$

(b) Set up an integral for the area of the region above the parabola and inside the ellipse.
Solution: Using symmetry in the y-axis,

$$2\int_0^2 \sqrt{1 - \frac{x^2}{4}} - \frac{1}{4}(x^2 - 4)\, dx$$

(c) Evaluate the integral in part (b).
Split the integral in two:

$$2\int_0^2 \sqrt{1 - \frac{x^2}{4}}\, dx - 2\int_0^2 \frac{1}{4}(x^2 - 4)\, dx$$

and do the second integral first, it's easier:

$$2\int_0^2 \sqrt{1 - \frac{x^2}{4}}\, dx - \left(\frac{x^3}{6} - 2x\right)\Big|_0^2 = \frac{8}{3} + 2\int_0^2 \sqrt{1 - \frac{x^2}{4}}\, dx$$

The simplest way to handle the first part is to interpret it geometrically and use the formula πab for the area of an ellipse to get a value of $\frac{1}{2}\pi(2)(1) = \pi$. Alternatively, the trig substitution $x = 2\sin\theta \Rightarrow dx = 2\cos\theta\, d\theta$ yields

$$2\int_0^2 \sqrt{1 - \frac{x^2}{4}}\, dx = 2\int_0^{\pi/2} \sqrt{1 - \sin^2\theta}2\cos\theta\, d\theta = 4\int_0^{\pi/2} \cos^2\theta\, d\theta = 4\int_0^{\pi/2} \frac{1}{2} + \frac{\cos 2\theta}{2}\, d\theta = \pi$$

Thus the total area in question is $\frac{8}{3} + \pi$.
You could also integrate over y:

$$2\int_{-1}^0 2\sqrt{y+1}\, dy + 2\int_0^1 2\sqrt{1 - y^2}\, dy = \frac{8}{3}(y+1)^{3/2}\Big|_{-1}^0 + \pi = \frac{8}{3} + \pi$$

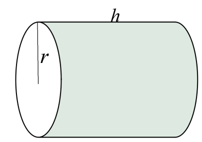
(d) Set up an integral for the volume generated when this region is revolved about the y-axis.
Solution: Integrating disks, we have

$$\pi\int_{-1}^1 r^2\, dy = \pi\int_{-1}^1 x^2\, dy = 4\pi\int_{-1}^0 y + 1\, dy + 4\pi\int_0^1 1 - y^2\, dy$$

Alternatively, shells produce

$$2\pi \int_0^2 rh\, dx = 2\pi \int_0^2 x \left(\sqrt{1 - \frac{x^2}{4}} - \frac{1}{4}x^2 + 1 \right) dx$$

(e) Evaluate the integral in part (d). **Solution:** $4\pi(\frac{1}{2}y^2 + y)\Big|_{-1}^{0} + 4\pi(y - \frac{1}{3}y^3)\Big|_0^1 = 2\pi + 4\pi - \frac{4\pi}{3} = \frac{14\pi}{3}$, or, if you used shells: $2\pi \int_0^2 x\sqrt{1 - \frac{x^2}{4}}\, dx - \frac{\pi}{8}x^4 + \pi x^2\Big|_0^2 = -4\pi \int_1^0 \sqrt{u}\, du + 2\pi = \frac{8\pi}{3}u^{3/2}\Big|_0^1 + 2\pi = \frac{14\pi}{3}$



2. (30 points) A cylinder or radius r and length h is placed horizontally on the ground as shown below.

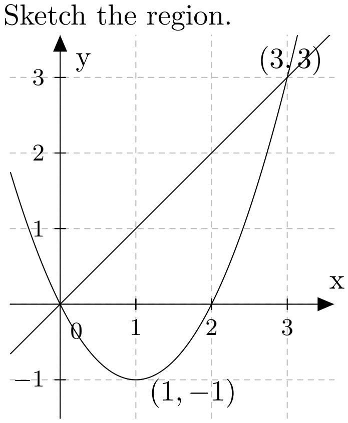
(a) Set up (but do not evaluate) an integral to find the total energy needed to pump water from the ground level to fill this empty cylinder. Assume the units are in meters and recall the force density of water is $9.8 \times 10^3 N/m^3$

Solution: Center your coordinate system, say, at the center of the circular cross-section and with the y-axis pointed upwards. Then a cross sectional area at an equipotential in the gravitational field is $A(y) = h \cdot \ell(y)$ where $\ell(y) = 2\sqrt{r^2 - y^2}$ is the cross-sectional length across the circle at coordinate y. Thus an infinitesimal piece of volume is $dV = A(y)dy = 2h\sqrt{r^2 - y^2}dy$ so an infinitesimal piece of force (weight) is $dF = 19600A(y)dy = 19600h\sqrt{r^2 - y^2}dy$. Now this force has to be lifted from the ground a distance $y - (-r) = y + r$ so the integral is $W = \int dW = 19600 \int_{-r}^{r} (y + r)h\sqrt{r^2 - y^2}dy$

(b) If $h = 10$ meters and $r = 2$ meters, what energy is generated by draining to the ground level the full cylinder to be half its height? **Solution:** $9.8 \times 10^3 h \int_0^r (y + 2)\sqrt{r^2 - y^2}dy = 9.8 \times 10^4 \int_0^2 (y + 2)\sqrt{4 - y^2}dy =$ $9.8 \times 10^4 \int_0^2 y\sqrt{4 - y^2}dy + 19.6 \times 10^4 \int_0^2 \sqrt{4 - y^2}dy$. The second integral is a quarter circle of radius 2, so that value is $1.96 \times 10^5\pi \approx 6.2 \times 10^5$Joules. For the first integral, substitute $u = 4 - y^2 \Rightarrow du = -2ydy$ so that integral is $\approx -3.2 \times 10^4 u^{3/2}\Big|_4^0 \approx 2.6 \times 10^5$Joules. So, all together, about 9×10^5 Joules.

3. (28 points) Consider the region bounded by $y = (x - 1)^2 - 1$ and $y = x$

(a) Sketch the region.



(b) Set up integral(s) for the volume generated by revolving this region about the y-axis using the "washer" method.

Solution: $\pi \int_0^3 \left(1 + \sqrt{y+1}\right)^2 - y^2\, dy$ $+\pi \int_{-1}^0 \left(1 + \sqrt{y+1}\right)^2 - \left(1 - \sqrt{y+1}\right)^2\, dy$

(c) Set up integral(s) for the volume generated by revolving this region about the y-axis using the "cylindrical shells" method.

Solution: $2\pi \int_0^3 x(x - (x - 1)^2 + 1)\, dx$

(d) Evaluate the above integrals to see they give the same value.

$$\pi \int_0^3 \left(1 + \sqrt{y+1}\right)^2 - y^2\, dy + \pi \int_{-1}^0 \left(1 + \sqrt{y+1}\right)^2 - \left(1 - \sqrt{y+1}\right)^2\, dy$$

$= \pi \int_0^3 2 + 2\sqrt{y+1} + y - y^2 \, dy + \pi \int_{-1}^0 2\sqrt{y+1} \, dy$

$= \pi(2y + \frac{4}{3}(y+1)^{3/2} + \frac{1}{2}y^2 - \frac{1}{3}y^3)\Big|_0^3 + \pi\frac{4}{3}(y+1)^{3/2}\Big|_{-1}^0 = \pi(6 + \frac{32}{3} + \frac{9}{2} - 9) + \frac{4\pi}{3} = \frac{27\pi}{2}$

The shells integral is $2\pi \int_0^3 x(x-(x-1)^2+1) \, dx = 2\pi \int_0^3 -x^3 + 3x^2 \, dx = 2\pi \left(-\frac{1}{4}x^4 + x^3\right)\Big|_0^3 = \pi\left(54 - \frac{81}{2}\right) = \frac{27\pi}{2}$

4. (12 points) Find a simplified function for the average value of $y = \sin x$ on the interval $[0, L]$ as a function of L.

Solution: $f_{avg} = \frac{1}{L}\int_0^L \sin x \, dx = \dfrac{1 - \cos L}{L}$