## Proj 1 Solutions

1. (15 points) Consider the function $f(x) = \sin\sqrt{x}$

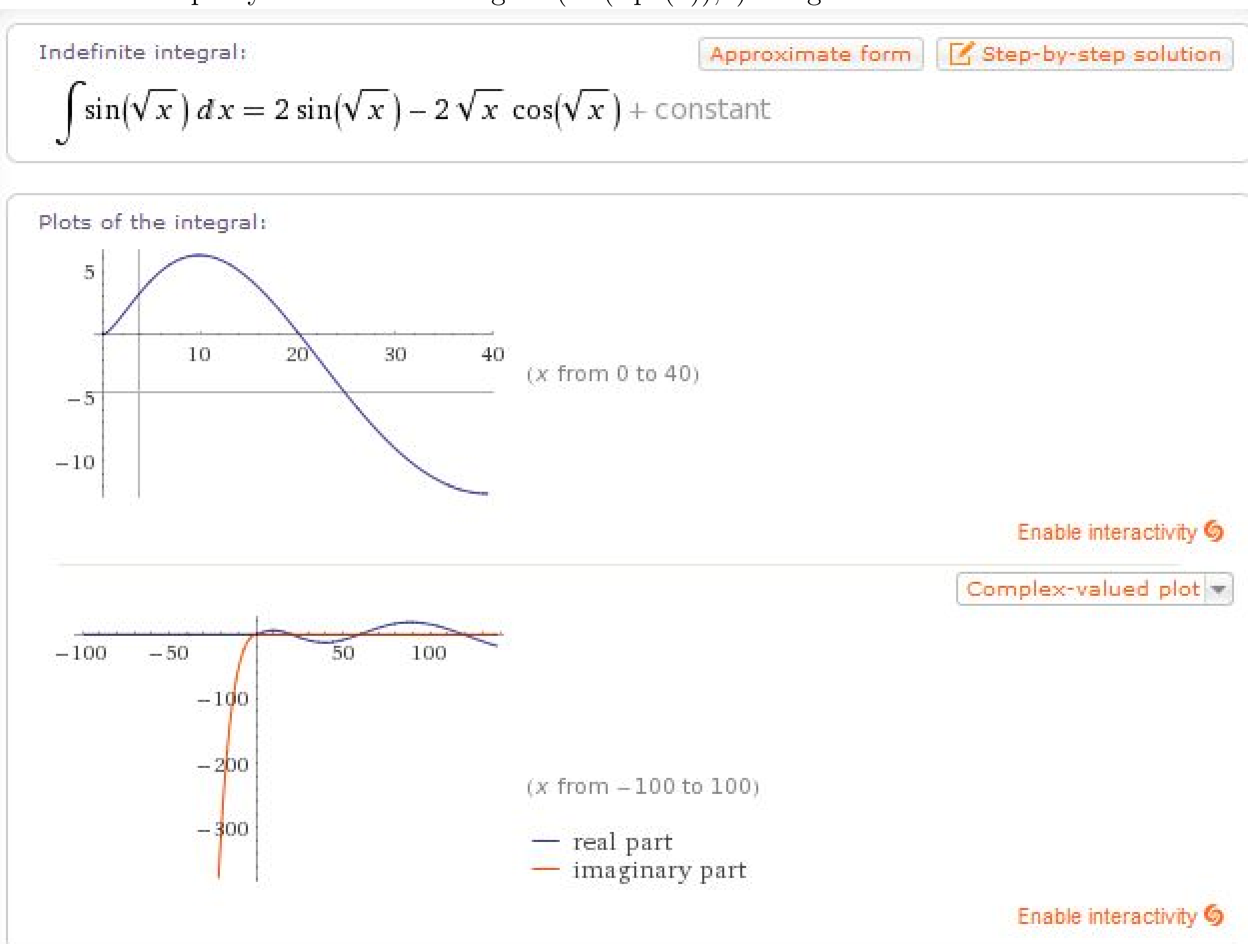
   (a) Use a CAS (such as Mathematica 9, available in the Math 4 Study Center) to simplify

   $$\int f(x)\,dx$$

   (in Mathematica, the command is `Integrate[Sin[Sqrt[x]], x]`). Verify that your result is correct by differentiating - do the differentiation by hand, showing steps.
   **Solution:** Using Mathematica, I get the result, 2 Sin[Sqrt[x]] - 2 Sqrt[x] Cos[Sqrt[x]].
   Using Derive (a CAS from 25 years ago) I get

   ```
   #1:  SIN(√x)
   #2:  ∫ SIN(√x) dx
   #3:                                  2·SIN(√x) - 2·√x·COS(√x)
   ```

   At Wolfram Alpha you can enter "integrate(sin(sqrt(x)),x)" to get even more detailed results:




   To be sure, we use the chain rule and the product rule to differentiate: $\frac{d}{dx}\left(2\sin\sqrt{x} - 2\sqrt{x}\cos(\sqrt{x})\right) = \frac{\cos\sqrt{x}}{\sqrt{x}} - \left(\frac{\cos\sqrt{x}}{\sqrt{x}} - \sin\sqrt{x}\right)$

(b) Use the FTC (part 2) to evaluate

$$\int_0^{\pi^2} \sin\sqrt{x}\,dx$$

.

**Solution:** $2\sin\sqrt{x} - 2\sqrt{x}\cos(\sqrt{x})\Big|_0^{\pi^2} = 2\sin\pi - 2\pi\cos\pi - (2\sin 0 - 0) = 2\pi$

(c) Use substitution to show that

$$\int_0^{\pi^2} \sin\sqrt{x}dx = 2\int_0^{\pi} u\sin u\,du$$

.

**Solution:** Let $u = \sqrt{x}$ so that $du = \frac{dx}{2\sqrt{x}} \Leftrightarrow dx = 2u\,du$. Also, $0 \le x \le \pi^2 \Leftrightarrow 0 \le u \le \pi$ Thus $\int_0^{\pi^2} \sin\sqrt{x}\,dx = \int_0^{\pi} \sin u(2u\,du)$ which is precisely what we want.

2. (15 points) What is the largest number of terms, $n$, that can be added so that $\sum_{k=1}^{n} k^2 < 1000$.

**Solution:** $\sum_{k=1}^{n} k^2 < 1000 \Leftrightarrow \dfrac{n(2n+1)(n+1)}{6} < 1000 \Leftrightarrow 2n^3 + 3n^2 + n - 6000 = 0$

Doing synthetic division and using the remainder theorem, we see that $n = 13$ is the largest value:

| | | | | |
|---|---|---|---|---|
| | 2 | 3 | 1 | − 6000 |
| 13 | | 26 | 377 | 4914 |
| | 2 | 29 | 378 | − 1086 |

| | | | | |
|---|---|---|---|---|
| | 2 | 3 | 1 | − 6000 |
| 14 | | 28 | 434 | 6090 |
| | 2 | 31 | 435 | 90 |

3. (20 points) Show step-by-step procedures for evaluating $\int_0^1 x^5\sqrt{x^2+1}\,dx$.

**Solution:** Let $u = x^2 + 1$ so that $du = 2x\,dx$ and $x^2 = u - 1 \Rightarrow x^4 = (u-1)^2$

Now as $0 \le x \le 1, 1 \le u \le 2$ Thus

$$\int_0^1 x^5\sqrt{x^2+1}\,dx = \frac{1}{2}\int_0^1 \sqrt{x^2+1}\cdot(x^4)\,(2xdx) = \frac{1}{2}\int_1^2 u^{1/2}\cdot(u-1)^2\,du =$$

$$= \frac{1}{2}\int_1^2 u^{5/2} - 2u^{3/2} + u^{1/2}\,du = \left(\frac{1}{7}u^{7/2} - \frac{2}{5}u^{5/2} + \frac{1}{3}u^{3/2}\right)\Big|_1^2 = \frac{u^{3/2}}{105}\left(15u^2 - 42u + 35\right)\Big|_1^2 =$$

$$\frac{2\sqrt{2}}{105}11 - \frac{8}{105} = \frac{22\sqrt{2}-8}{105}$$

4. (30 points) Approximate the integral

$$\int_1^3 x^4\,dx$$

(a) by sampling left endpoints of 10 equal subintervals ($L_{10}$)

**Solution:** $\Delta x = \dfrac{3-1}{10} = \dfrac{1}{5}$ so $x_i = 1 + \dfrac{i}{5}$ whence $L_{10} \approx \frac{1}{5}\sum_{i=1}^{10}\left(1 + \dfrac{i-1}{5}\right)^4 = \frac{1}{5}\sum_{i=1}^{10}\left(\dfrac{i+4}{5}\right)^4$

$$= \frac{625+1296+2401+4096+6561+10000+14641+20736+28561+38416}{3125} = \frac{127333}{3125} \approx 40.766$$

(b) by sampling right endpoints of 10 equal subintervals ($R_{10}$)

**Solution:** $R_{10} \approx \frac{1}{5}\sum_{i=1}^{10}\left(1+\frac{i}{5}\right)^4 = \frac{1}{5}\sum_{i=1}^{10}\left(\frac{i+5}{5}\right)^4$

$$= \frac{1296+2401+4096+6561+10000+14641+20736+28561+38416+50625}{3125} = \frac{177333}{3125} \approx 56.747$$

(c) by averaging left and right endpoint sums.

**Solution:** $\dfrac{40.766+56.747}{2} \approx 48.76$

(d) Which of these approximating sums are overestimates? Why?

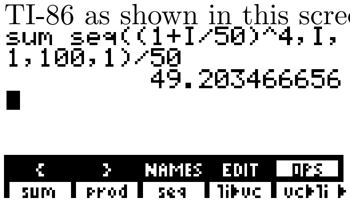
**Solution:** Since the function is increasing on $[1,3]$, $L_{10} < A < R_{10}$

Since the function is concave up, the average of $L_10$ and $R_{10}$, which is the trapezoidal area, will be greater than the actual area on the curve.

(e) Repeat the above with $n = 100$.

**Solution:** $\Delta x = \dfrac{3-1}{100} = \dfrac{1}{50} \Rightarrow x_i = 1 + \dfrac{i}{50}$

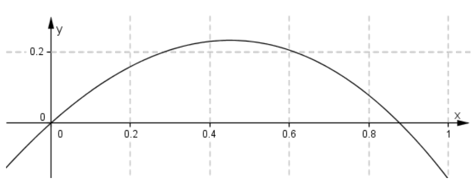
$L_{100} = \dfrac{1}{50}\sum_{i=1}^{100}\left(1+\dfrac{i-1}{50}\right)^4$ In Mathematica write `Sum[(1 + i/50)∧4, i, 0, 99]/50` to get $\frac{1487608333}{31250000}$ and then approximate with `N[%]` for 47.6035. Alternatively, use the `list` menu on the TI-86 as shown in this screen shot:

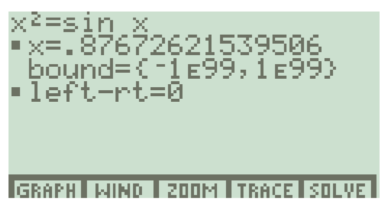



The trapezoid is the average: $\dfrac{47.603+49.203}{2} \approx 48.398$

5. (20 points) Approximate to the nearest thousandth the area that lies above the $x$-axis and below

(a) $y = \sin x - x^2$.

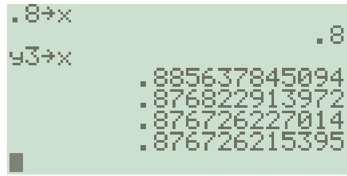



**Solution:** The region where $0 \le y \le \sin x - x^2$ is shown. The upper bound requires more computing power. There is no closed-form solution to the equation $x^2 = \sin x$. To get a good approximation, some technology is helpful. We could use the TI-86's solver function, as shown above right or you could use Newton's method for a zero of $f(x) = x^2 - \sin x$ by iterating

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - \sin x_n}{2x_n - \cos x_n}$$

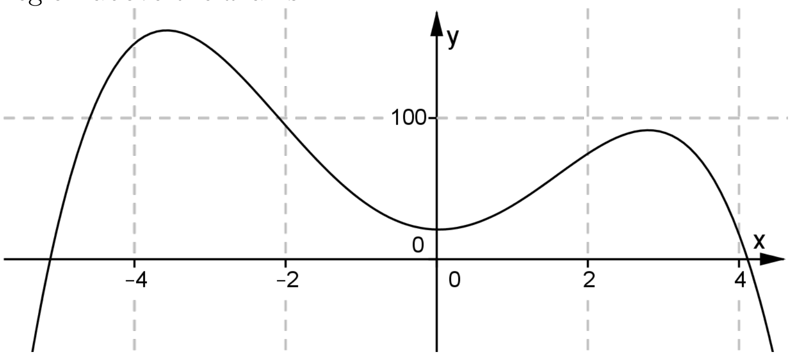
starting at, say, $x_0 = 0.8$. You can set this up on the TI-86 by putting `y1=x∧2-sin x`, `y2=der1(y1,x)`, and `y3=x-y1/y2`, as shown in the screenshot on the left below. Then put 0.8 in x (the `STO` command) and then the output of `y3` back into x, and iterated as shown in the screen shot on the right below:


Then the area is $A \approx \int_0^{0.8767262154} \sin x - x^2\,dx = \left(-\cos x - \frac{1}{3}x^3\right)\Big|_0^{0.8767262154} =$ $-\cos(0.8767262154) - \frac{1}{3}(0.8767262154)^3 - (-1) \approx 0.136$, which seems reasonable.

(b) $y = 21 - x + 20x^2 - x^3 - x^4$

**Solution:** This is similar to the previous problem. The graph shows the curve bounds a single region above the $x$-axis:




Before resorting to technology, a bit of tinkering with the function shows that if we split up the square term, it factors by grouping:

$$f(x) = 21 - x - x^2 + 21x^2 - x^3 - x^4 = (21 - x - x^2) + x^2(21 - x - x^2) = -(x^2+1)(x^2+x-21)$$

The irreducible factor $x^2+1$ has not real roots and the roots of $x^2+x-21$ are the bounds of our integral: $\frac{1}{2}\left(-1-\sqrt{85}\right) \le x \le \frac{1}{2}\left(-1+\sqrt{85}\right)$ So we can compute the area exactly:

$$A = \int_{\frac{1}{2}\left(-1-\sqrt{85}\right)}^{\frac{1}{2}\left(-1+\sqrt{85}\right)} 21 - x + 20x^2 - x^3 - x^4\,dx = \left(21x - \frac{1}{2}x^2 + \frac{20}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5\right)\Bigg|_{\frac{1}{2}\left(-1-\sqrt{85}\right)}^{\frac{1}{2}\left(-1+\sqrt{85}\right)}$$

$= \frac{935\sqrt{85}}{12} \approx 718.356$, where I let Mathematica work out the details at the end. Interesting that the rational part of the result is zero.