Math 1B - Spring '14 - Final Exam Solutions 5/22/14

1. The graph below plots the velocity of a bicycle accelerating downhill over a period of one minute. Points from the graph are tabulated here:

t (min)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
v (m/sec)	0	0.13	0.36	0.66	1.01	1.41	1.86	2.34	2.86	3.42	4.00

Approximate the distance traveled by the bicyclist during this minute with

(a) 5 subintervals of equal lengths and right endpoints as sample points.

Solution: $\Delta t = \dfrac{1-0}{5} = 0.2 \Rightarrow x_i = a + i\Delta t = 0.2i$. We need to convert 0.2 min = 12 sec, so $R_5 = \Delta t \sum_{i=1}^{5} v(0.2i) = 12(0.36 + 1.01 + 1.86 + 2.86 + 4) = 12 \cdot 10.09 = 121.08$ meters.

(b) 5 trapezoids of equal widths.

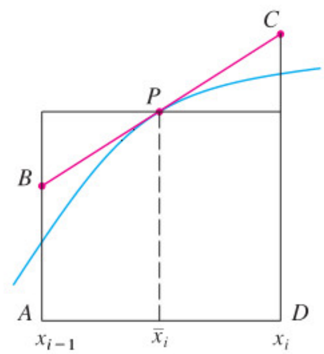
Solution: The trapezoidal sum is computed by averaging left and right sums. $L_5 = \Delta t \sum_{i=1}^{5} v(0.2i) = 12(0 + 0.36 + 1.01 + 1.86 + 2.86) = 12 \cdot 6.09 = 73.08$ meters, so $T_5 = \dfrac{121.08 + 73.08}{2} = 97.08$

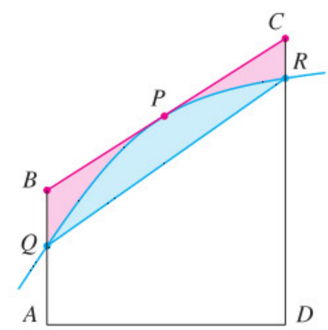
(c) 5 subintervals of equal lengths and midpoints as sample points.

Solution: $M_5 = 12(0.13 + 0.66 + 1.41 + 2.34 + 3.42) = 12 \cdot 7.96 = 95.52$ meters

(d) Explain the relationship between the trapezoidal sum and the midpoint sum in terms of the concavity of the curve.

Solution: Since the curve is concave up the midpoint approximation will be less than the true value which will, in turn, be less than the trapezoid sum. The picture below is from the text (section 7.7, page 509). In the picture, the curve is concave up, so the opposite conclusion results, but the idea is the same.





2. Evaluate:

(a) $\int_0^9 \frac{d}{dx}\sqrt{271 + x^3}\, dx = \sqrt{271 + x^3}\big|_0^9 = 10\sqrt{10} - \sqrt{271}$

(b) Let $u = \cos x$ so $\dfrac{d}{dx}\int_0^{\cos(x)} \sin^{-1} t\, dt = \dfrac{du}{dx}\dfrac{d}{du}\int_0^u \sin^{-1} t\, dt = -\sin x \sin^{-1}(\cos x) = \left(x - \dfrac{\pi}{2}\right)\sin x$

3. Use integration by parts and substitution methods to evaluate the integral.

Recall that $\frac{d}{dy}\sin^{-1} y = \dfrac{1}{\sqrt{1-y^2}}$

(a) $\int_0^1 \sin^{-1} y\, dy = y\sin^{-1} y\Big|_0^1 - \int_0^1 \dfrac{y\, dy}{\sqrt{1-y^2}} = \dfrac{\pi}{2} - \int_0^{\pi/2} \dfrac{\sin\theta\, \cos\theta d\theta}{\cos\theta} = \dfrac{\pi}{2} + \left(\cos\theta\Big|_0^{\pi/2}\right) = \dfrac{\pi}{2} - 1 \approx 0.57$

$u = \sin^{-1} y$	$dv = dy$
$du = \dfrac{dy}{\sqrt{1-y^2}}$	$v = y$

$y = \sin\theta$
$dy = \cos\theta d\theta$

(b) $\int_0^1 \left(\sin^{-1} y\right)^2 dy = y\left(\sin^{-1} y\right)^2\Big|_0^1 - \int_0^1 2\sin^{-1} y \dfrac{y\, dy}{\sqrt{1-y^2}} dy = \dfrac{\pi^2}{4} - 2\int_0^{\pi/2} w \sin w\, dw = \cdots = \dfrac{\pi^2}{4} - 2 \approx 0.47$

$u = \left(\sin^{-1} y\right)^2$	$dv = dy$
$du = 2\sin^{-1} y \dfrac{dy}{\sqrt{1-y^2}}$	$v = y$

$w = \sin^{-1} y \Leftrightarrow y = \sin w$
$dw = \dfrac{dy}{\sqrt{1-y^2}}$

$x = w$	$dz = \sin w\, dw$
$dx = dw$	$z = -\cos w$

4. Consider the integral function,

$$I_n(t) = \int_0^t \sec^n(x)\,dx = \int_0^t \sec^{n-2}(x)\sec^2(x)\,dx$$

(a) Use integration by parts to show that

$$I_n(t) = \frac{\tan t \sec^{n-2} t}{n-1} + \frac{n-2}{n-1} I_{n-2}(t)$$

Make a table of u, dv, du, and v. Look for the integral recurring after applying a trig identity.

Solution: $I_n(t) = \int_0^t \sec^{n-2}(x)\sec^2(x)\,dx = \sec^{n-2}(x)\tan(x)\Big|_0^t - (n-2)\int_0^t \sec^{n-2}(x)\tan^2(x)\,dx =$

$u = \sec^{n-2}(x)$	$dv = \sec^2(x)\,dx$
$du = (n-2)\sec^{n-2}(x)\tan(x)\,dx$	$v = \tan(x)$

$= \sec^{n-2}(t)\tan(t) - (n-2)\int_0^t \sec^{n-2}(x)(\sec^2(x)-1)\,dx = \sec^{n-2}(t)\tan(t) - (n-2)(I_n(t) - I_{n-2}(t))$

$\Leftrightarrow (n-1)I_n(t) = \sec^{n-2}(t)\tan(t) + (n-2)I_{n-2}(t)$ ..and divide both sides by $n-1$ for the result.

(b) Use the reduction formula from (a) to evaluate $\int_0^{\pi/4} \sec^4(x)\,dx$

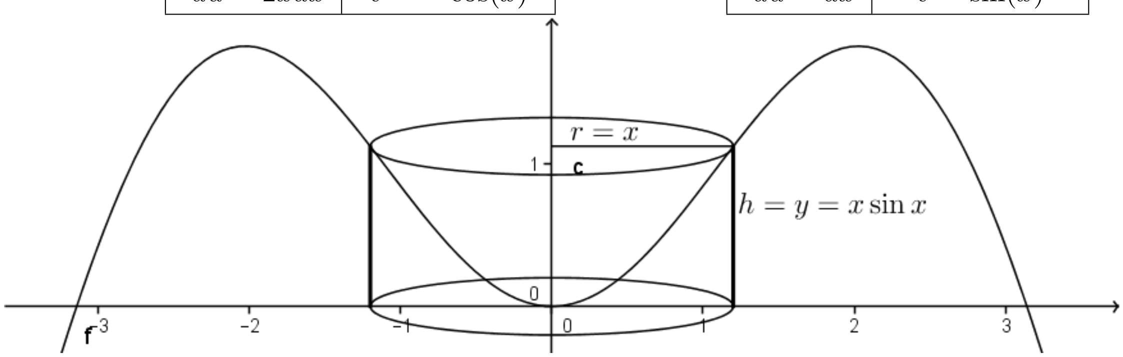
Solution: $I_4\left(\frac{\pi}{4}\right) = \dfrac{\tan\left(\frac{\pi}{4}\right)\sec^2\left(\frac{\pi}{4}\right)}{3} + \frac{2}{3}I_2\left(\frac{\pi}{4}\right) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$

5. The region bounded by the x-axis and the curve $y = x\sin x$ between $x = 0$ and $x = \pi$ is revolved around the y-axis. Find the volume generated.
Solution: This is most easily done with shells, I think. An element of volume is $dV = 2\pi rh dx = 2\pi x^2 \sin x\,dx$ so that the volume is $V = 2\pi\int_0^\pi x^2 \sin x\,dx$ which will require two integration by parts:

$$V = -2\pi x^2\cos x\Big|_0^\pi + 4\pi\int_0^\pi x\cos x\,dx = 2\pi^3 + 4\pi x\sin x\Big|_0^\pi - 4\pi\int_0^\pi \sin x\,dx = 2\pi^3 + 4\pi\cos x\Big|_0^\pi = 2\pi^3 - 8\pi$$

$u = x^2$	$dv = \sin(x)\,dx$
$du = 2xdx$	$v = -\cos(x)$

$u = x$	$dv = \cos(x)\,dx$
$du = dx$	$v = \sin(x)$



6. The ellipse bounded by the parametric equations $\begin{array}{c} x = \cos t \\ y = 2\sin t \end{array}$ is revolved around the y-axis and the volume generated is filled with water. Set up and evaluate an integral to find the minimum work required to pump the water out through a hole in the top. Recall that the force density of water is approximately 9800 Newtons per cubic meter.
Solution:

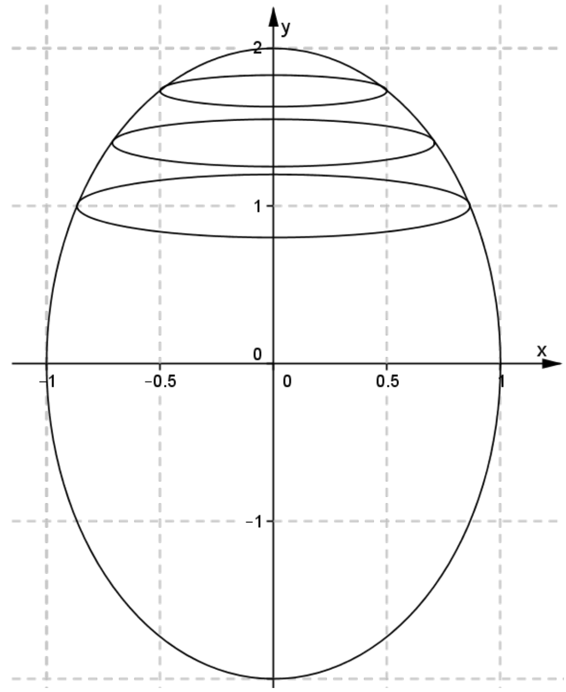
Cross-sectional areas at an equipotential in the gravitational field are circles of radius $x = \cos(t)$ whose elements of volume are $dV = \pi x^2 dy = \pi\cos^2(t)(2\cos(t)dt)$. Multiplying by the force density of water gives an element of force, $dF = 9800(2\cos^3(t)dt)$ which must be moved a distance $2 - y = 2 - 2\sin(t)$, which requires work

$$W = 19600\pi \int_{-\pi/2}^{\pi/2} \cos^3(t)(2 - 2\sin(t))dt =$$

$$39200\pi \left[\int_{-\pi/2}^{\pi/2} \cos(t)(1-\sin^2(t))dt - \int_{-\pi/2}^{\pi/2} \cos^3(t)\sin(t)dt \right]$$

$$= 39200\pi \left[\sin(t) - \tfrac{1}{3}\sin^3(t) \Big|_{-\pi/2}^{\pi/2} - 0 \right]$$

$$= 39200\pi \left(1 - \tfrac{1}{3} - \left(-1 + \tfrac{1}{3}\right)\right) = \frac{156800\pi}{3} \text{ Joules}$$



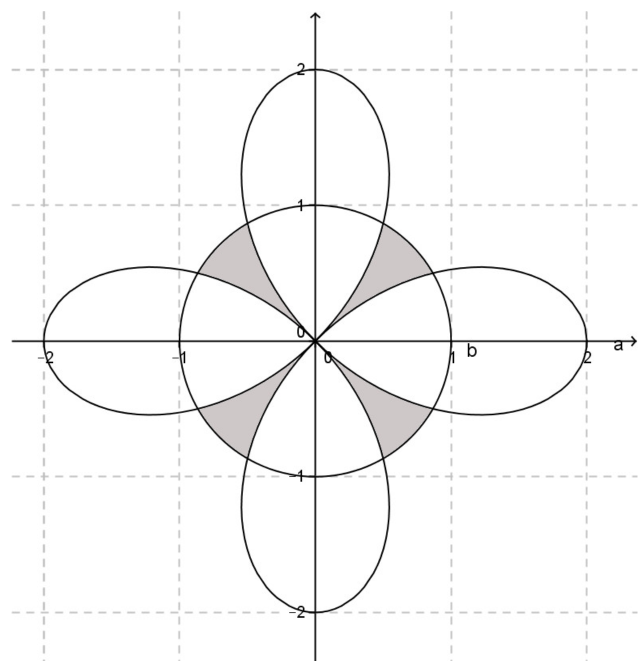
7. Find the area of the region outside the polar curve $r = 2\cos 2\theta$ and inside the unit circle, as shaded in the diagram.
Solution: The curves intersect where $2\cos 2\theta = 1$
$\Leftrightarrow 2\theta = \pm\frac{\pi}{3} + 2\pi k$ where $k \in \mathbb{Z} \Leftrightarrow \theta = \pm\frac{\pi}{6} + k\pi$

Using symmetry, $A = 8 \int_{\pi/6}^{\pi/4} \frac{1}{2} - \frac{4\cos^2 2\theta}{2} d\theta$

$$= 8 \int_{\pi/6}^{\pi/4} -\frac{1}{2} - \cos 4\theta d\theta =$$

$$= -4\left(\frac{\pi}{4} - \frac{\pi}{6}\right) - 2\left(\sin 4\theta \Big|_{\pi/6}^{\pi/4}\right)$$

$$= \sqrt{3} - \frac{\pi}{3}$$



8. Use a Maclaurin series to approximate each to the nearest billionth (10^{-9}).

(a) $\sin(0.1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{10^{2n+1}(2n+1)!} = \frac{1}{10} - \frac{1}{6000} + \frac{1}{12,000,000} + \cdots \approx 0.0998\overline{3}$

(b) $\sqrt[3]{1.1} = (1+0.1)^{1/3} = \sum_{n=0}^{\infty} \binom{1/3}{n}(0.1)^n = 1 + \frac{1}{30} - \frac{1}{900} + \frac{5}{81000} - \frac{1}{243000} + \frac{22}{72,900,000} - \cdots$

$= 1.0\overline{3} - \frac{1}{900} + \frac{5}{81000} - \frac{1}{243000} + \frac{22}{72,900,000} - \cdots = 1.03\overline{2} + \frac{1}{16200} - \frac{1}{243000} + \frac{22}{72,900,000} - \cdots =$

$1.032\overline{283950617} - \frac{1}{243000} + \frac{22}{72,900,000} - \cdots \approx 1.0322798 \approx 1.0322801$

9. Find the 6th degree Taylor polynomial approximation for $\sin^2(x)$ in two ways. These should be the same.

(a) by squaring the Taylor polynomial for $\sin(x)$

Solution: $\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)^2 \approx x^2 - \frac{x^4}{3} + \left(\frac{1}{60} + \frac{1}{36}\right) x^6 = x^2 - \frac{x^4}{3} + \frac{2}{45}x^6$

(b) by using $\sin^2 x = \frac{1 - \cos 2x}{2}$ and expanding $\cos 2x$ in a Taylor polynomial.

Solution: $\sin^2 x = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right) \approx x^2 - \frac{x^4}{3} + \frac{2}{45}x^6$