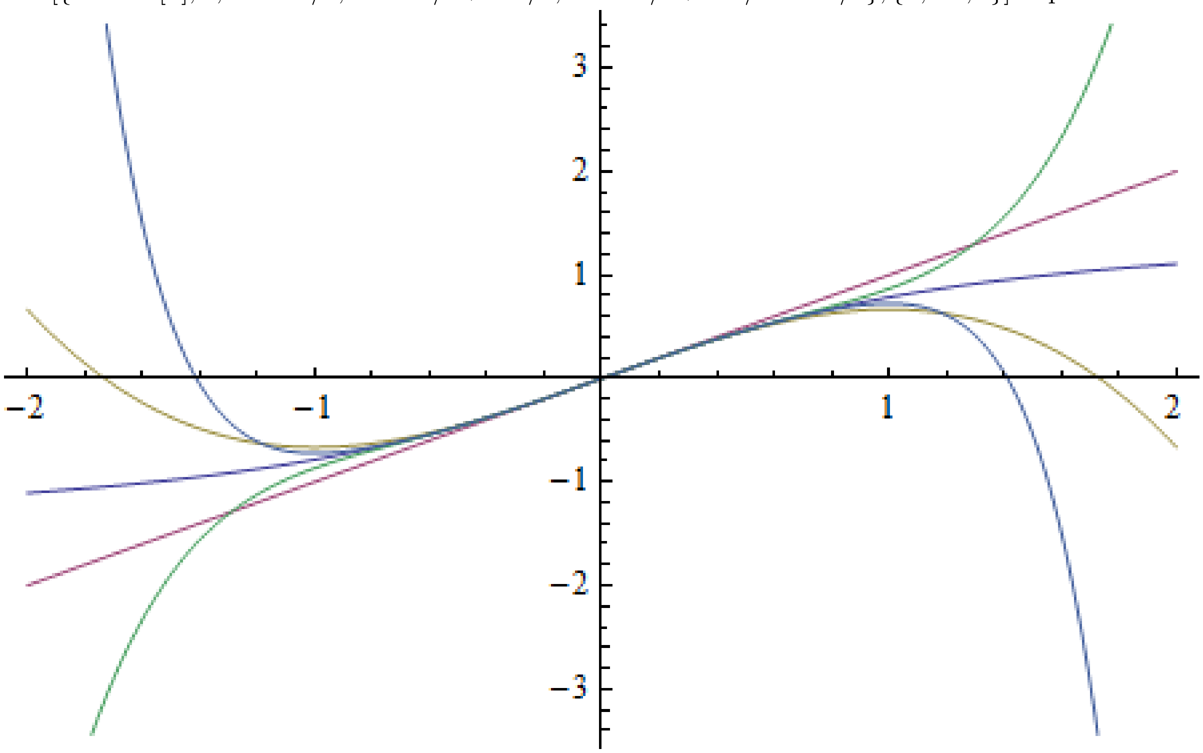
Chapter 11 Notes

1. Here's some work we did in class using series to approximate π, as a recap.

Recall the derivation of $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ on$[-1,1]$ is obtained by differentiating $y = \arctan x$, expanding as a geometric series and then integrating.
To be sure, the polynomials diverge outside this interval, as, using the Mathematica command Plot[{ArcTan[x], x, $x-x$^3/3, $x-x$^3/3+x^5/5, $x-x$^3/3+x^5/5$-x$^7/7}, {x, -2, 2}] to produce the diagram illustrates:



Note that $\arctan x$ is the one that appears to be following the asymptotes, $y = \pm\frac{\pi}{2}$ and that the curve that veers off most severely is also the curve that best fits the function on $[-1,1]$ – that's the 7th degree polynomial.

A beginner might then use the series $\frac{\pi}{4} = \arctan(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, but that converges slowly.

We can improve the rate of convergence. Use the double angle formula starting with $\alpha = \arctan\frac{1}{5}$ so that $\tan(2\alpha) = \frac{2\cdot\tan\alpha}{1-\tan^2\alpha} = \frac{2/5}{1-1/25} = \frac{5}{12}$ and then applying the double angle formula again: $\tan(4\alpha) = \frac{2\cdot\tan(2\alpha)}{1-\tan^2(2\alpha)} = \frac{5/6}{1-25/144} = \frac{120}{119}$

Since $\frac{\pi}{4} \approx 4\alpha$, compute $\tan\left(4\alpha - \frac{\pi}{4}\right) = \frac{\tan 4\alpha - \tan\pi/4}{\tan 4\alpha + \tan\pi/4} = \frac{\tan 4\alpha - 1}{1+\tan 4\alpha} = \frac{1}{239}$ so that $4\alpha - \pi/4 = \arctan(1/239)$ and

$$\pi = 16\cdot\arctan(1/5) - 4\cdot\arctan(1/239) = 16\sum_{n=0}^{\infty}\frac{(-1)^n(1/5)^{2n+1}}{2n+1} + 4\sum_{n=0}^{\infty}\frac{(-1)^n(1/239)^{2n+1}}{2n+1} =$$

$$= 16\left(\frac{1}{5} - \frac{1}{375} + \frac{1}{15625} + \cdots\right) - 4\left(\frac{1}{219} - \frac{1}{31510377} + \cdots\right)$$
$$= 3.2 - 0.042\overline{6} + 0.001024 + \cdots - 0.\overline{01826484} + 1.27\times 10^{-7}$$
$$= 3.158357\overline{3} - 0.\overline{01826484} + 1.27\times 10^{-7}$$
≈ 3.1400 – With four terms we get three digits of accuracy...or so it seems.

2. 11.3 # 39: Estimate $S = \sum_{n=1}^{\infty} (2n+1)^{-6}$ correct to five decimal places.

Solution: Since $f(x) = \dfrac{1}{(2x+1)^6}$ is a positive, decreasing, integrable function on $[1, \infty)$, we can use the approximation fro the the error in approximation: $R_N = S - S_N < \int_N^{\infty} \dfrac{dx}{(2x+1)^6} = \lim_{b\to\infty} \dfrac{-1}{10(2x+1)^5}\Big|_N^b = \dfrac{1}{10(2N+1)^5}$. Choose N so that this is smaller than the max rounding error for 10^{-5}, that is, so that $\dfrac{1}{10(2N+1)^5} < 5 \times 10^{-6} \Leftrightarrow \dfrac{1}{(2N+1)^5} < 5 \times 10^{-5} \Leftrightarrow (2N+1)^5 > 2 \times 10^4$ or $2N+1 > (20000)^{1/5} \approx 7.25 \Leftrightarrow N > 3$. So we'll need four terms. To be sure, here's a table of partial sums:

N	1	2	3	4	5
S_N	0.00137174	0.00143574	0.00144424	0.00144612	0.00144669

where you can see that 0.001446 would round up to it's final approximation, 0.00145 on the 4th iteration, but not before.

3. 11.4 # 27 Determine whether the series $\sum_{n=1}^{\infty} \left(1 + \dfrac{1}{n}\right)^2 e^{-n}$ converges or diverges.

Solution: Comparison test? Compare with what? The first factor is approaching 1 as n grows, but the second factor is decreasing geometrically. Try comparison with a geometric series.
Since $1 + \dfrac{1}{n} < 3$ for $n \geq 1$ $\sum_{n=1}^{\infty} \left(1 + \dfrac{1}{n}\right)^2 e^{-n} < \sum_{n=0}^{\infty} 3e^{-n} = \dfrac{3}{1 - 1/e} = \dfrac{3e}{e-1}$.
The series has all positive terms which are less than those of a convergent series, thus the series is convergent.

4. 11.4 # 31 Determine whether the series $\sum_{n=1}^{\infty} \sin\left(\dfrac{1}{n}\right)$ converges or diverges.

Solution: This is a classic limit comparison test problem. All terms are positive, and since $\lim_{n\to\infty} \dfrac{\sin(1/n)}{1/n} = \lim_{x\to 0} \dfrac{\sin x}{x} = 1$. The harmonic series diverges, so, by limit comparison, $\sum_{n=1}^{\infty} \sin\left(\dfrac{1}{n}\right)$ also diverges.

5. 11.4 # 33 Use the sum of the first 10 terms to approximate the sum of the series $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n^4+1}}$. Estimate the error.

Solution: This requires a electro-mechanical computer of some sort. Mathematica will take this command:
`Sum[1/Sqrt[b^4+1],{n,10}]` and spit back $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{17}} + \frac{1}{\sqrt{82}} + \frac{1}{\sqrt{257}} + \frac{1}{\sqrt{626}} + \frac{1}{\sqrt{1297}} + \frac{1}{\sqrt{2402}} + \frac{1}{\sqrt{4097}} + \frac{1}{\sqrt{6562}} + \frac{1}{\sqrt{10001}}$ which is approximated by `N[%]` for 1.24856. We can estimate the error through a mixture of comparison and integral test. $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n^4+1}} < \sum_{n=1}^{\infty} \dfrac{1}{n^2}$ and the error is then less than $\int_{10}^{\infty} \dfrac{dx}{x^2} = \dfrac{1}{10}$. In fact, summing the first 10000 terms gives 1.34362, so 1/10 is a pretty good estimate of the error.