Math 1B—Calculus II – Chapters 8 and 10 Problems, Fall 2011 Name______________________________
Write all responses on separate paper. Show your work for credit. Don't use a calculator.

1. Find the length of the curve $x = \frac{y^3}{6} + \frac{1}{2y}$ between $y = 1$ and $y = 2$.

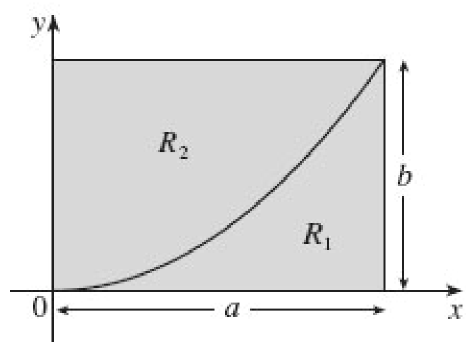
2. Find the length of the curve described by the parametric equations $x = e^t + e^{-t}$, $y = 2t - 4$ for $0 \le t \le 2$.

3. Find the surface area generated by rotating about the y–axis the curve $y = \frac{1}{1+x^2}$ for $1 \le x \le 2$.

4. Consider the surface area generated by revolving about the x-axis the portion of the curve $r = f(\theta)$ between where $\theta = a$ and $\theta = b$.
   a. Show that this surface area is computed by $\int_a^b 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$
   b. Use this formula to show that when the curve $r = 2\sin\theta$ for $0 \le \theta \le \frac{\pi}{2}$ is revolved about the x-axis, the surface area generated is 4π.

5. Suppose the lifetime (in days) of an electric light bulb is a random variable with a probability density function, $L(t) = \begin{cases} 0 & \text{if } t < 0 \\ kte^{-\pi t} & \text{if } t \ge 0 \end{cases}$
   a. Find the value of k that is needed for $L(t)$ to be a probability density function.
   b. Find the mean value for the lifetime of this electric light bulb.
   c. Express the probability that this light bulb lasts more than 100 days as an integral.

6. A rectangle R with sides a and b is divided into two parts, R_1 and R_2, by the curve $y = x^3$, as shown in the figure at right. Find the centroid of each part.



7. The ellipse with major axis of length 4 and minor axis of length 2 is submerged in water so that its major axis is horizontal and the center of the ellipse is 100 meters below the surface. Set up but do not evaluate an integral to compute the total fluid force on one face of the ellipse.

Math 1B—Calculus II – Chapters 8 and 10 Problems Solutions

1. Find the length of the curve $x = \frac{y^3}{6} + \frac{1}{2y}$ between $y = 1$ and $y = 2$.

   SOLN: $1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{y^2}{2} - \frac{1}{2y^2}\right)^2 = 1 + \left(\frac{y^4}{4} - \frac{1}{2} + \frac{1}{4y^4}\right) = \frac{y^4}{4} + \frac{1}{2} + \frac{1}{4y^4} = \left(\frac{y^2}{2} + \frac{1}{2y^2}\right)^2$

   Thus $L = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^2 \left(\frac{y^2}{2} + \frac{1}{2y^2}\right) dy = \frac{y^3}{6} - \frac{1}{2y}\Big|_1^2 = \frac{4}{3} - \frac{1}{4} - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{17}{12}$

2. Find the length of the curve described by the parametric equations $x = e^t + e^{-t}$, $y = 2t - 4$ for $0 \le t \le 2$.

   SOLN: $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t - e^{-t})^2 + 4 = e^{2t} - 2 + e^{-2t} + 4 = e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2$

   Thus $L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^2 (e^t + e^{-t})\, dt = e^t - e^{-t}|_0^2 = e^2 - \frac{1}{e^2}$

3. Find the surface area generated by rotating about the y–axis the curve $y = \frac{1}{1+x^2}$ for $1 \le x \le 2$.

   SOLN: $\int 2\pi r ds = 2\pi \int_1^2 x\sqrt{1 + (y')^2}dx = 2\pi \int_1^2 x\sqrt{1 + \left(\frac{-2x}{(1+x^2)^2}\right)^2}\, dx = 2\pi \int_1^2 \frac{x\sqrt{(x^2+1)^4+4x^2}}{1+x^2} dx$

   It seems this integral doesn't have an elementary antiderivative. The command,
   Integrate[$x * \mathrm{Sqrt}[(x^2 + 1)^4 + 4x^2] / (1 + x^2), x$] just returns the unsimplified integral,

   $\int_1^2 \frac{x\sqrt{(x^2+1)^4+4x^2}}{x^2+1} dx \approx 5.43992$

4. Consider the surface area generated by revolving about the y-axis the portion of the curve $r = f(\theta)$ between where $\theta = a$ and $\theta = b$.

   a. Show that this surface area is computed by $\int_a^b 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

   SOLN: Using $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$ we have

   $\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$ and $\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$ so that

   $$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2$$

   $$= \left(\frac{dr}{d\theta}\right)^2 (\cos^2\theta + \sin^2\theta) + 2r\frac{dr}{d\theta}(\sin\theta\cos\theta - \cos\theta\sin\theta) + r^2(\sin^2\theta + \cos^2\theta)$$

   Thus $ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

   and the surface area of revolution @ x-axis is $\int 2\pi y ds = \int_a^b 2\pi r\sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

   b. Use this formula to show that when the curve $r = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$ is revolved about the x-axis, the surface area generated is 4π.

   SOLN: $2\pi \int_0^{\pi/2} (2\cos\theta)\sin\theta \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}d\theta = 8\pi \int_0^{\pi/2} \cos\theta\sin\theta\, d\theta =$
   $4\pi(\sin\theta)_0^{\pi/2} = 4\pi$

5. Suppose the lifetime (in days) of an electric light bulb is a random variable with a probability density function , $L(t) = \begin{cases} 0 & \text{if } t < 0 \\ kte^{-\pi t} & \text{if } t \geq 0 \end{cases}$

   a. Find the value of k that is needed for $L(t)$ to be a probability density function.

   SOLN: $\int_{-\infty}^{\infty} L(t)dt = \int_0^{\infty} kte^{-\pi t}dt = \lim_{b\to\infty} \int_0^b kte^{-\pi t}dt = \ldots$

   use integration by parts with $\begin{pmatrix} u = kt & dv = e^{-\pi t}dt \\ du = kdt & v = -\frac{1}{\pi}e^{-\pi t} \end{pmatrix} \ldots$

$$= \lim_{b\to\infty} -\frac{kt}{\pi}e^{-\pi t}\Big|_0^b + \frac{k}{\pi}\int_0^b e^{-\pi t}dt$$

$$= \lim_{b\to\infty} \frac{-bk}{\pi e^{\pi b}} + 0 - \frac{k}{\pi^2}e^{-\pi t}\Big|_0^b = \frac{k}{\pi^2} \text{ (using L'Hopital's rule.)}$$

   So for L to qualify as a pdf, we require that $k = \pi^2$.

   b. Find the mean value for the lifetime of this electric light bulb.

   SOLN: The mean life time is $\int_{-\infty}^{\infty} tL(t)dt = \pi^2 \int_0^{\infty} t^2e^{-\pi t}dt = \lim_{b\to\infty} \pi^2 \int_0^b t^2e^{-\pi t}dt =$

   … do parts twice - first with $\begin{pmatrix} u = \pi^2t^2 & dv = e^{-\pi t}dt \\ du = 2\pi^2tdt & v = -\frac{1}{\pi}e^{-\pi t} \end{pmatrix} \ldots = \lim_{b\to\infty} -\frac{\pi t^2}{e^{\pi t}}\Big|_0^b + 2\pi\int_0^b te^{-\pi t}dt$

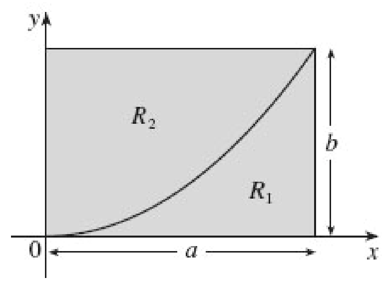
   $= 0 + \lim_{b\to\infty} 2\pi \int_0^b te^{-\pi t}dt$. Then use $\begin{pmatrix} u = 2\pi t & dv = e^{-\pi t}dt \\ du = 2\pi dt & v = -\frac{1}{\pi}e^{-\pi t} \end{pmatrix}$ so that

$$\int_{-\infty}^{\infty} tL(t)dt = \lim_{b\to\infty}\left(-2te^{-\pi t}|_0^b + 2\int_0^b e^{-\pi t}dt\right) = 0 + 2\left(\lim_{b\to\infty} -\frac{1}{\pi}e^{-\pi b} + \frac{1}{\pi}\right) = \frac{2}{\pi}$$

   c. Express the probability that this light bulb lasts more than 100 days as an integral.

   SOLN: $\pi^2 \int_{100}^{\infty} e^{-\pi t}dt$

6. A rectangle R with sides a and b is divided into two parts, R_1 and R_2, by the curve $y = x^3$, as shown in the figure at right. Find the centroid of each part. SOLN: Note that $a^3 = b$ in this diagram.



For R_1, the area is $\int_0^a x^3dx = \frac{a^4}{4}$ and so the area of $R_2 = ab - \frac{a^4}{4} = \frac{3a^4}{4}$.

For R_1,

$$M_x = \int_0^b y\left(a - \sqrt[3]{y}\right)dy = \frac{ay^2}{2} - \frac{3y^{\frac{7}{3}}}{7}\Bigg|_0^{a^3} = \frac{a^7}{2} - \frac{3a^7}{7} = \frac{a^7}{14}.$$

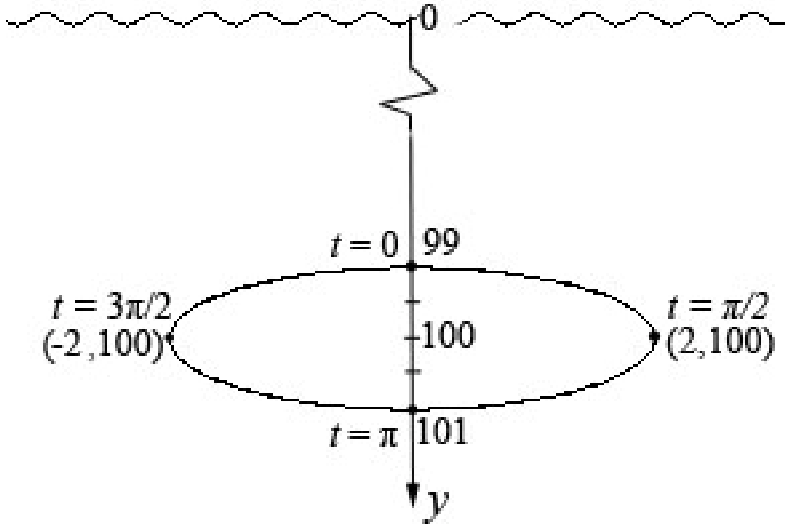
Also $M_Y = \int_0^a x \cdot x^3dx = \frac{a^5}{5}$ so for R_1, $(\overline{x}, \overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{a^5}{5}\cdot\frac{4}{a^4}, \frac{a^7}{14}\cdot\frac{4}{a^4}\right) = \left(\frac{4a}{5}, \frac{2a^3}{7}\right)$.

For R_2, $M_Y = \int_0^a x \cdot (a^3 - x^3)\,dx = \frac{a^3x^2}{2} - \frac{x^5}{5}\Big|_0^a = \frac{3a^5}{10}$ and

$$M_x = \int_0^b y\left(\sqrt[3]{y}\right)dy = \frac{3}{7}(a^3)^{\frac{7}{3}} = \frac{3a^7}{7}$$

so for R_2, $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{3a^5}{10}\cdot\frac{4}{3a^4}, \frac{3a^7}{7}\cdot\frac{4}{3a^4}\right). = \left(\frac{2a}{5}, \frac{4a^3}{7}\right)$

7. The ellipse with major axis of length 4 and minor axis of length 2 is submerged in water so that its major axis is horizontal and the center of the ellipse is 100 meters below the surface. Set up but do not evaluate an integral to compute the total fluid force on one face of the ellipse.



SOLN: Center the coordinate system at the surface of the water directly above the center of the ellipse with the y-axis increasing in the downward direction. Then the ellipse can be parameterized as

$$x = 2\sin t \text{ and } y = 100 - \cos t.$$

Then the four extreme points of the ellipse correspond to $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, like so:

$$\left(x(0), y(0)\right) = (0{,}99)$$
$$\left(x\left(\frac{\pi}{2}\right), y\left(\frac{\pi}{2}\right)\right) = (2{,}100)$$
$$\left(x(\pi), y(\pi)\right) = (0{,}101)$$
$$\left(x\left(\frac{3\pi}{2}\right), y\left(\frac{3\pi}{2}\right)\right) = (-2{,}100)$$

Infinitesimal fluid force = force density of water * depth * infinitesimal area

$$= 9810y(2x)dy = 9810(100 - \cos t)(4\sin t)(\sin t\, dt)$$

Integrating from the top to the bottom (as $t = 0$ to π) so have the total fluid force is

$$F = \int dF = 39240 \int_0^{\pi} (100 - \cos t)\sin^2 t\, dt$$

Of course you can also set this up using rectangular coordinates. If we center the ellipse at the origin and orient the y-axos to increase upwards, then the equation for the ellipse is $\frac{x^2}{4} + y^2 = 1$. An infinitesimal element of fluid force is then

$$9810(100 - y)2xdy = 19620(100 - y)2\sqrt{1 - y^2}dy$$

So the total fluid force is $F = 39240 \int_{-1}^{1} (100 - y)\sqrt{1 - y^2}dy$