Math 1B – Calculus II – Fall '11 – Chapter 7 Problems Name________________________

Write all responses on separate paper. Show your work for credit. Don't use a calculator.

1. Find the volume generated when the region bounded by $y = \sin x$ and $y = 0$ between $x = 0$ and $x = \pi$ is revolved around the y-axis. Recall that an element of volume is given by the shell $dV = 2\pi rhdx$.

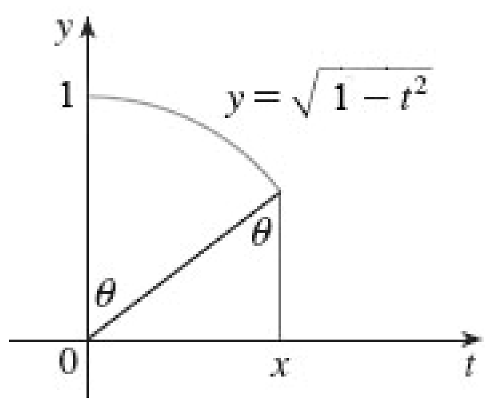
2. Use integration by parts to find a reduction formula for $\int_0^1 x^{2n} e^{3x}\,dx$.

3. Evaluate the integral $\int_0^{\pi/6} \cos(3x)\cos(6x)\,dx$ by
   a. Using the identity $\cos 2\theta = 2\cos^2\theta - 1$
   b. Using the identity $\cos A \cos B = ½\,[\cos(A - B) + \cos(A + B)]$

4. A particle moves along a straight line with velocity $v(t) = \cos(\omega t)\sin^2(\omega t)$. Find its position function $s = f(t)$ if $f(0) = 1$.

5. Find $\int \frac{1}{x^2 - a}\,dx$ in terms of a if
   a. $a < 0$
   b. $a > 0$

6. Use trigonometric substitution to prove that
   $$\int_0^x \sqrt{1-t^2}\,dt = \frac{1}{2}\arcsin(x) + \frac{x}{2}\sqrt{1-x^2}$$
   then interpret both terms on the right side of the equation in terms of areas in the figure at right.



7. Find the average value of $f(x) = x^4\sqrt{4 - x^2}$ on the interval [0,2].

8. Show the proper way of evaluating the improper integral $\int_1^\infty \frac{1}{x^2+1}\,dx$

9. Determine whether the improper integral converges or diverges. Use comparison, if necessary.
   a. $\int_1^\infty \frac{1}{\sqrt{x^2+1}}\,dx$
   b. $\int_2^\infty \frac{1}{\sqrt{x^3+1}}\,dx$

Math 1B – Calculus II – Fall '11 – Chapter 7 Problem Solutions

1. Find the volume generated when the region bounded by $y = \sin x$ and $y = 0$ between $x = 0$ and $x = \pi$ is revolved around the y-axis. Recall that an element of volume is given by the shell $dV = 2\pi rhdx$.

   SOLN: Using $\boxed{\begin{array}{ll} u = x & dv = \sin x\,dx \\ du = dx & v = -\cos x \end{array}}$

   $$\int dV = 2\pi \int_0^\pi x \sin x\, dx = -2\pi x \cos x|_0^\pi + 2\pi \int_0^\pi \cos x\, dx = 2\pi^2 + 0 + 0 - 0 = 2\pi^2$$

2. Use integration by parts to find a reduction formula for $\int_0^1 x^{2n} e^{3x} dx$.

   SOLN: Using $\boxed{\begin{array}{ll} u = x^{2n} & dv = e^{3x}dx \\ du = 2nx^{2n-1}dx & v = \frac{1}{3}e^{3x} \end{array}}$

   $$\int_0^1 x^{2n} e^{3x} dx = \frac{1}{3} x^{2n} e^{3x}\Big|_0^1 - \frac{2n}{3}\int_0^1 x^{2n-1} e^{3x} dx = \frac{e^3}{3} - \frac{2n}{3}\int_0^1 x^{2n-1} e^{3x} dx$$

3. Evaluate the integral $\int_0^{\pi/6} \cos(3x)\cos(6x)\,dx$ by

   a. Using the identity $\cos 2\theta = 2\cos^2\theta - 1$

   SOLN:

   $$\begin{aligned} \int_0^{\pi/6} \cos(3x)\left(2\cos^2(3x) - 1\right) dx &= \int_0^{\pi/6} 2\cos^3(3x)\,dx - \int_0^{\pi/6} \cos(3x)\,dx \\ &= 2\int_0^{\pi/6} \cos(3x)\left(1 - \sin^2 3x\right) dx - \int_0^{\pi/6} \cos(3x)\,dx \\ &= \int_0^{\pi/6} \cos(3x)\,dx - 2\int_0^{\pi/6} \cos(3x)\sin^2 3x\,dx \end{aligned}$$

   (Let $u = \sin(3x)$ so that $du = 3\cos(3x)\,dx$)

   $$= \frac{\sin 3x}{3}\Big|_0^{\pi/6} - \frac{2u^3}{9}\Big|_0^1 = \frac{1}{3} - \frac{2}{9} = \frac{1}{9}$$

   b. Using the identity $\cos A \cos B = ½\,[\cos(A - B) + \cos(A + B)]$

   SOLN:

   $$\int_0^{\pi/6} \cos(3x)\cos(6x)\,dx = \frac{1}{2}\int_0^{\pi/6} \cos(-3x) + \cos(9x)\,dx = \frac{\sin(3x)}{6}\Big|_0^{\pi/6} + \frac{\sin(9x)}{18}\Big|_0^{\pi/6} = \frac{1}{6} - \frac{1}{18} = \frac{1}{9}$$

4. A particle moves along a straight line with velocity $v(t) = \cos(\omega t)\sin^2(\omega t)$. Find its position function $s = f(t)$ if $f(0) = 1$.

   SOLN: $s = 1 + \int_0^t \cos(\omega u)\sin^2(\omega u)du$. Substituting $v = \sin(\omega u)$ so that $dv = \omega\cos(\omega u)du$, we have

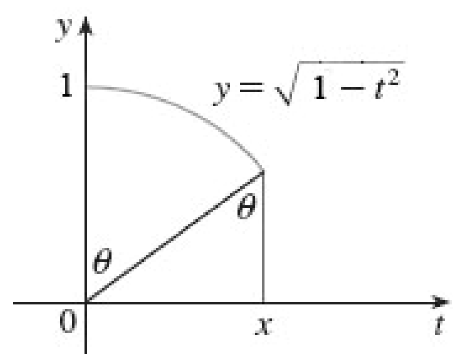
   $$s = 1 + \int_0^t \cos(\omega u)\sin^2(\omega u)du = 1 + \frac{1}{\omega}\int_0^{\sin(\omega t)} v^2 dv = 1 + \frac{v^3}{3\omega}\Big|_0^{\sin(\omega t)} = 1 + \frac{\sin^3(\omega t)}{3\omega}$$

5. Find $\int \frac{1}{x^2 - a} dx$ in terms of a if

a. $a < 0$
If $a < 0$ then substitute $x^2 = -a\tan^2\theta$ so that $x = \sqrt{-a}\tan\theta$ and $dx = \sqrt{-a}\sec^2\theta\, d\theta$ and
$$\int \frac{1}{x^2-a} dx = \sqrt{-a}\int \frac{\sec^2\theta d\theta}{-a\tan^2\theta - a} = -\frac{\sqrt{-a}}{a}\int d\theta = -\frac{\sqrt{-a}}{a}\theta + c =$$
$$-\frac{\sqrt{-a}}{a}\arctan\frac{x}{\sqrt{-a}} + c$$

b. $a > 0$
If $a > 0$ then we can factor the denominator to get
$$\int \frac{1}{x^2-a} dx = \int \frac{1}{(x-\sqrt{a})(x+\sqrt{a})} dx = \frac{\sqrt{a}}{2a}\int \frac{1}{x-\sqrt{a}} - \frac{1}{x+\sqrt{a}} dx$$
$$= \frac{\sqrt{a}}{2a}\left(\ln\left(x-\sqrt{a}\right) - \ln\left(x+\sqrt{a}\right)\right)$$
Note that this can also be done with the substitution, $x = \sqrt{a}\sec\theta$ so that $dx = \sqrt{a}\sec\theta\tan\theta\, d\theta$ and the integral becomes $\int \frac{1}{x^2-a} dx = \int \frac{\sqrt{a}\sec\theta\tan\theta}{a\sec^2\theta - a} d\theta = \int \frac{\sqrt{a}\sec\theta\tan\theta}{a\tan^2\theta} d\theta =$
$$\int \frac{\sqrt{a}}{a\sin\theta} d\theta = -\frac{\sqrt{a}}{a}\ln(\csc\theta + \cot\theta) + c = \frac{-\sqrt{a}}{a}\ln\left(\frac{x}{\sqrt{x^2-a}} + \frac{\sqrt{a}}{\sqrt{x^2-a}}\right) + c,$$
which is another, equivalent, expression.

6. Use trigonometric substitution to prove that $\int \sqrt{1-t^2}dt = \frac{1}{2}\arcsin(x) + \frac{x}{2}\sqrt{1-x^2}$ then interpret both terms on the right side of the equation in terms of areas in the figure.



SOLN: The area in the figure is $\int_0^x \sqrt{1-t^2}dt$
Let $t = \sin\theta$ so that $dt = \cos\theta\, d\theta$ and we have
$$\int_0^x \sqrt{1-t^2}dt = \int_0^{\arcsin x} \sqrt{1-\sin^2\theta}\cos\theta\, d\theta =$$
$$\int_0^{\arcsin x} \cos^2\theta\, d\theta = \int_0^{\arcsin x} \frac{1+\cos(2\theta)}{2} d\theta =$$
$$\left.\frac{\theta}{2} + \frac{\sin(\theta)\cos(\theta)}{2}\right|_0^{\arcsin x} = \frac{1}{2}\arcsin(x) + \frac{x}{2}\sqrt{1-x^2}$$
In the diagram, $\theta = \arcsin(x)$, so the area of the unit circle sector with central angle θ is $\frac{\theta r^2}{2} = \frac{1}{2}\arcsin(x)$ and the area of the triangular region is ½ base*height $= \frac{x}{2}\sqrt{1-x^2}$

7. Find the average value of $f(x) = x^4\sqrt{4-x^2}$ on the interval [0,2].

SOLN: Average value is $\frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2}\int_0^2 x^4\sqrt{4-x^2}dx$. Let $x = 2\sin(\theta)$ so that the integral is $\frac{1}{2}16\int_0^{\pi/2}\sin^4\theta\sqrt{4-4\sin^2\theta}\,2\cos\theta\,d\theta = 32\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta =$

$32\int_0^{\pi/2}(\sin\theta\cos\theta)^2\sin^2\theta\,d\theta = 32\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2\frac{(1-\cos 2\theta)}{2}d\theta =$

$4\int_0^{\pi/2}\sin^2 2\theta\,(1-\cos 2\theta)d\theta = 4\int_0^{\pi/2}\sin^2 2\theta\,d\theta - 4\int_0^{\frac{\pi}{2}}\sin^2 2\theta\cos 2\theta\,d\theta =$

$4\int_0^{\frac{\pi}{2}}\frac{1-\sin 4\theta}{2}d\theta - \int_0^0 u^2du = 2\theta + \frac{1}{2}\cos 4\theta\Big|_0^{\frac{\pi}{2}} = \pi$

8. Show the proper way of evaluating the improper integral $\int_1^\infty \frac{1}{x^2+1}dx$

SOLN: $\int_1^\infty \frac{1}{x^2+1}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2+1}dx = \lim_{b\to\infty}\arctan x\Big|_1^b =$
$\lim_{b\to\infty}\arctan b - \arctan 1 = \frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

9. Determine whether the improper integral converges or diverges. Use comparison, if necessary.

a. $\int_1^\infty \frac{1}{\sqrt{x^2+1}}dx = \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{x^2+1}}dx$ Substitute $x=\tan\theta$, $dx=\sec^2\theta\,d\theta$ so that

$\int_1^\infty \frac{1}{\sqrt{x^2+1}}dx = \lim_{b\to\infty}\int_{\pi/4}^{\arctan b}\frac{\sec^2\theta}{\sec\theta}d\theta =$

$= \lim_{b\to\infty}\ln\left(b+\sqrt{b^2+1}\right) - \ln\left(1+\sqrt{2}\right) = \infty$ so the integral is

divergent…but that's hard. Why not observe that if $x > 1$ then $\frac{1}{\sqrt{x^2+1}} > \frac{1}{\sqrt{x^2+x^2}} = \frac{\sqrt{2}}{2x} > 0$

so that $\int_1^\infty \frac{1}{\sqrt{x^2+1}}dx > \frac{\sqrt{2}}{2}\int_1^\infty \frac{1}{x}dx = \infty$?

b. $\int_1^\infty \frac{1}{\sqrt{x^3+1}}dx$ SOLN: Note that $\frac{1}{\sqrt{x^3+1}} < \frac{1}{\sqrt{x^3}} = x^{-\frac{3}{2}}$ so that $\int_1^\infty \frac{1}{\sqrt{x^3+1}}dx < \int_1^\infty x^{-\frac{3}{2}}dx$

$= \lim_{b\to\infty}\int_1^b x^{-\frac{3}{2}}dx = \lim_{b\to\infty} -2x^{-\frac{1}{2}}\Big|_1^b = \lim_{b\to\infty} 2-\frac{2}{\sqrt{b}} = 2$, so the integral is convergent.