Math 1B –Calculus – Chapter 11 Problems – Fall '11 Name___________________________________

Show work for credit. Write all responses on separate paper. No calculators.

1. Consider the infinite series $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$
   a. Explain how you know this is a convergent series.
   b. Use integrals to show why the error in approximating $S \approx S_9 = \sum_{n=1}^{9} \frac{1}{n^2}$ is between $\frac{1}{10}$ and $\frac{1}{9}$.
   c. Find a value of n that will ensure that the error in the approximation $S \approx S_n$ is less than $\frac{1}{500}$.

2. Determine whether the series is convergent or divergent. Justify your answer.
   a. $\sum_{n=1}^{\infty} \frac{n^2}{1+n^3}$
   b. $\sum_{n=1}^{\infty} 9 \cos\frac{1}{n}$
   c. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+n^3}$

3. Find a power series representation for the function. Find the radius and interval of convergence for each.
   a. $f(x) = \frac{1}{1-x^2}$
   b. $g(x) = \frac{x}{5x^2+1}$

4. Use differentiation or integration to find a power series for the function.
   a. $f(x) = \frac{1}{(3+x)^2}$
   b. $L(x) = \ln(1 + x^2)$

5. Consider $f(x) = \frac{10}{\sqrt[4]{1+x}}$
   a. Expand $f(x)$ as a power series.
   b. Use the first three non-zero terms of the series to approximate $f\left(\frac{1}{5}\right)$ and use the first neglected term to approximate the error in approximation.

6. Approximate $f(x) = e^{3x^2}$ by a Taylor polynomial of degree 4, expanded about $x = 0$.

7. Find the first 3 non-zero terms of $f(x) = e^x \cos x$, expanded about $x = 0$. *Hint*: Multiply the Maclaurin series and combine like terms.

Math 1B –Calculus – Chapter 11 Problems Solutions

1. Consider the infinite series $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$
   a. Explain how you know this is a convergent series.
   SOLN: It's a p-series with p > 1, therefore it's convergent.
   Also, the terms are all positive and decreasing so the integral test yields,
   $\int_1^{\infty} \frac{1}{x^2} dx = \lim_{k\to\infty} -\frac{1}{x}\Big|_1^k = 0 - (-1) = 1$. Since the integral converges, the series converges.
   b. Use integrals to show why the error in approximating $S \approx S_9 = \sum_{n=1}^{9} \frac{1}{n^2}$ is between $\frac{1}{10}$ and $\frac{1}{9}$.
   SOLN: $\int_{10}^{\infty} \frac{1}{x^2} dx = \lim_{k\to\infty} -\frac{1}{x}\Big|_{10}^k = \frac{1}{10} < S - S_9 < \int_9^{\infty} \frac{1}{x^2} dx = \lim_{k\to\infty} -\frac{1}{x}\Big|_9^k = \frac{1}{9}$.
   c. Find a value of n that will ensure that the error in the approximation $S \approx S_n$ is less than $\frac{1}{500}$.
   SOLN: $S - S_N < \int_N^{\infty} \frac{1}{x^2} dx = \lim_{k\to\infty} -\frac{1}{x}\Big|_N^k = \frac{1}{N} \le \frac{1}{500}$ if N ≥ 500
2. Determine whether the series is convergent or divergent. Justify your answer.
   a. $\sum_{n=1}^{\infty} \frac{n^2}{1+n^3}$
   SOLN: $\frac{n^2}{1+n^3} \ge \frac{n^2}{n^3+n^3} = \frac{1}{2n}$, thus $\sum_{n=1}^{\infty} \frac{n^2}{1+n^3} > \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series diverges, or p-series with p = 1.) Therefore the series diverges by comparison.
   b. $\sum_{n=1}^{\infty} 9\cos\frac{1}{n}$
   SOLN: Here $\lim_{n\to\infty} 9\cos\frac{1}{n} = 9 \neq 0$, so the series diverges by the *n*th term test.

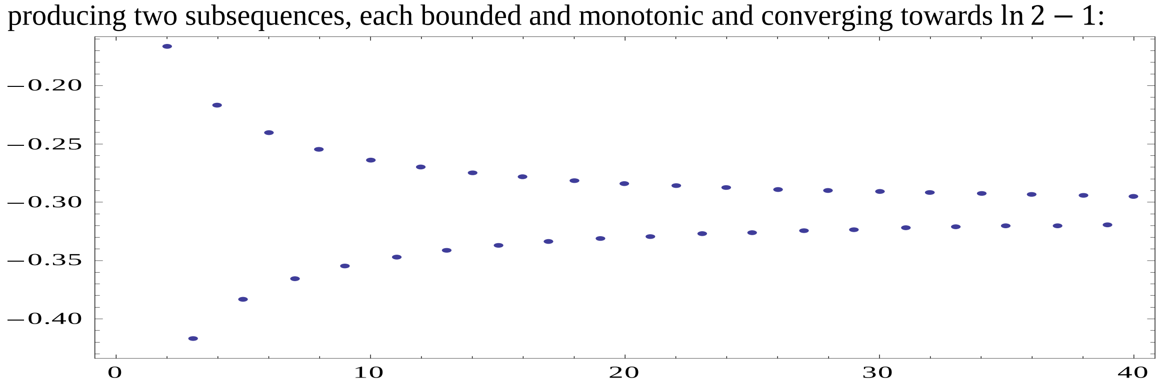
c. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+n^3}$. Note this is more simply expressed as $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+1}$

SOLN: This is an alternating series with decreasing terms and $\lim_{n\to\infty} \frac{1}{n+1} = 0$, therefore it is convergent.

Furthermore $\ln(1+x) = \int \frac{dx}{1+x} = \int \frac{1}{1-(-x)} dx = \int \sum_{n=0}^{\infty} (-x)^n \, dx = \sum_{n=0}^{\infty} (-1)^n \int x^n dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$

So $\ln 2 = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{n+1} \Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{1}{n+1} = (\ln 2) - 1 \approx -0.30685281944$

The Mathematica Command ListPlot[Table[{n, Sum[(−1)^i ∗ (i + 1)^−1, {i, 1, n}]}, {n, 1,40}], Frame → True, Axes → False] Produces a plot of the first 40 partial sums…you can see them as a damped oscillating producing two subsequences, each bounded and monotonic and converging towards $\ln 2 - 1$:



3. Find a power series representation for the function. Find the radius and interval of convergence for each.

a. $f(x) = \frac{1}{1-x^2}$

SOLN: $\frac{1}{1-x^2} = \sum_{n=0}^{\infty}(x^2)^n = \sum_{n=0}^{\infty} x^{2n}$ The radius of convergence is 1 and the interval of convergence is $(-1,1)$.

b. $g(x) = \frac{x}{5x^2+1}$

SOLN: $\frac{x}{5x^2+1} = x \cdot \frac{1}{1-(-5x^2)} = x \cdot \sum_{n=0}^{\infty}(-5x^2)^n = \sum_{n=0}^{\infty}(-1)^n 5^n x^{2n+1}$

The ratio test has $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5^{n+1}x^{2n+3}}{5^n x^{2n+1}}\right| = 5x^2 < 1 \Leftrightarrow -\frac{\sqrt{5}}{5} < x < \frac{\sqrt{5}}{5}\ x$ So the radius of convergence is $\frac{\sqrt{5}}{5}$. If $x = \pm\frac{\sqrt{5}}{5}$ then $\sum_{n=0}^{\infty}(-1)^n 5^n x^{2n+1} = \frac{\sqrt{5}}{5}\sum_{n=0}^{\infty}(-1)^n 5^n \left(\pm\frac{\sqrt{5}}{5}\right)^{2n}$ $= \frac{\sqrt{5}}{5}\sum_{n=0}^{\infty}(-1)^n 5^n \left(\frac{1}{5}\right)^n = \frac{\sqrt{5}}{5}\sum_{n=0}^{\infty}(-1)^n$ is divergent. Thus the interval of convergence is $\left(-\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)$

4. Use differentiation or integration to find a power series for the function.

a. $f(x) = \frac{1}{(3+x)^2}$

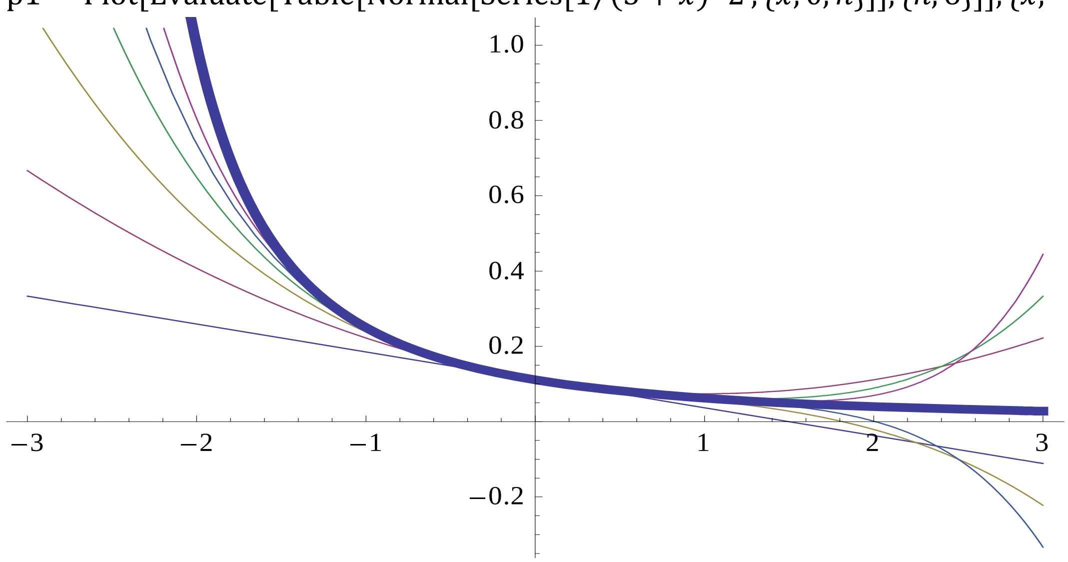
SOLN: $\frac{1}{(3+x)^2} = -\frac{1}{3}\frac{d}{dx}\left(\frac{1}{1-\left(-\frac{x}{3}\right)}\right) = -\frac{1}{3}\frac{d}{dx}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^{n+1}\frac{d}{dx}x^n$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^{n+1} n x^{n-1} = \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^{n+2}(n+1)x^n$$

To be sure, the Mathematica command, Series[1/(3 + x)^2 , {x, 0,6}], produces the first 6 terms of the series: $\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27} - \frac{4x^3}{243} + \frac{5x^4}{729} - \frac{2x^5}{729} + \frac{7x^6}{6561} + O(x^7)$.

Furthermore, the sequence of approximating polynomials can be graphed in the interval of convergence $(-3,3]$ using

p1 = Plot[Evaluate[Table[Normal[Series[1/(3 + x)^2 , {x, 0, n}]], {n, 6}]], {x, −3,3}]



b. $L(x) = \ln(1+x^2)$

SOLN: $\ln(1+x^2) = \int \frac{2x}{1+x^2}dx = 2\int x \cdot \frac{1}{1-(-x^2)}dx =$

$$2\int x \cdot \sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty}(-1)^n 2\int x^{2n+1}dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{n+1}$$

5. Consider $f(x) = \frac{10}{\sqrt[4]{1+x}}$

a. Expand $f(x)$ as a power series.

SOLN: $10(1+x)^{\frac{-1}{4}} = 10\sum_{n=0}^{\infty}\binom{-1/4}{n}x^n$. As to $\binom{-1/4}{n} = \frac{\left(\frac{-1}{4}\right)\left(\frac{-5}{4}\right)\left(\frac{-9}{4}\right)\cdots\left(\frac{-1}{4}-n+1\right)}{n!}$

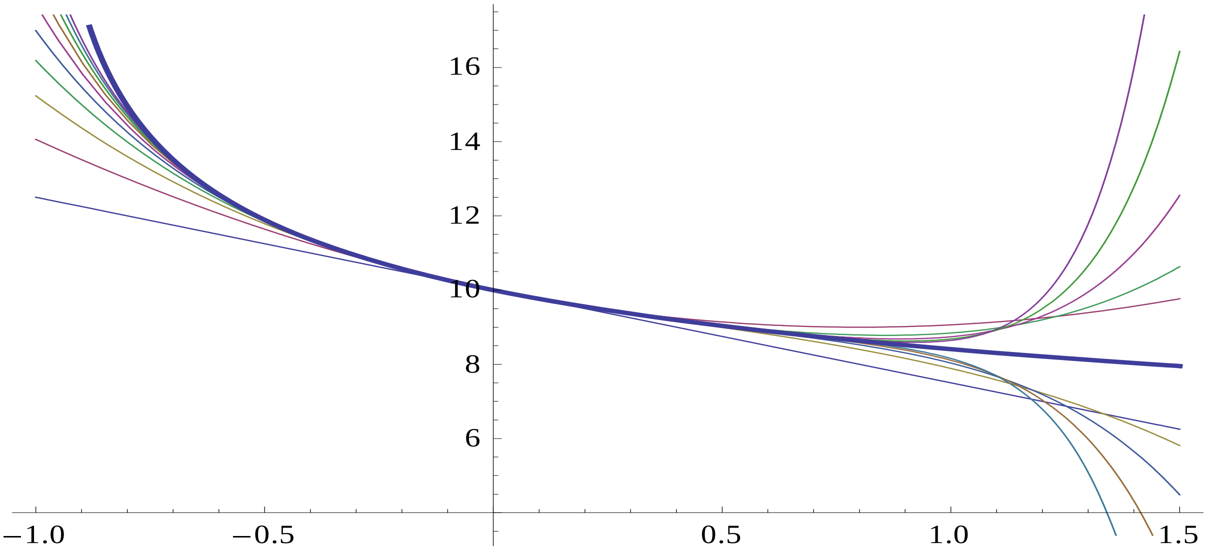
b. Use the first three non-zero terms of the series to approximate $f\left(\frac{1}{5}\right)$ and use the first neglected term to approximate the error in approximation.

SOLN: $\frac{10}{\sqrt[4]{1+x}} \approx 10\left(1-\frac{1}{4}x+\frac{5}{32}x^2\right)$ so that $\frac{10}{\sqrt[4]{1+1/5}} \approx 10-\frac{1}{2}+\frac{1}{16} = \frac{153}{16}$

The error in approximation is less than $10\left(\frac{45}{128}\right)\left(\frac{1}{5^3}\right) = \frac{9}{320}$

As an afterthought, we can use the Mathematica command, Series$\left[10/(1+x)^{1/4},\{x,0,6\}\right] =$ $10-\frac{5x}{2}+\frac{25x^2}{16}-\frac{75x^3}{64}+\frac{975x^4}{1024}-\frac{3315x^5}{4096}+\frac{23205x^6}{32768}+O(x^7)$ for the first 7 terms, and plot these:

Plot[Evaluate[Table[Normal[Series[10 * (1 + x)^(−1/4), {x, 0, n}]], {n, 10}]], {x, −1,1.5}]



6. Approximate $f(x) = e^{3x^2}$ by a Taylor polynomial of degree 4, expanded about $x = 0$.

SOLN: $e^{3x^2} = \sum_{n=0}^{\infty}\frac{(3x^2)^n}{n!} \approx 1+3x^2+\frac{9x^4}{2}$

7. Find the first 3 non-zero terms of $f(x) = e^x\cos x$, expanded about $x = 0$. *Hint*: Multiply the Maclaurin series and combine like terms.

SOLN: $e^x \cos x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right) \approx 1 + x - \frac{1}{3}x^3$

Using the Mathematica commands,

$$\text{P1} = \text{Plot}\left[\text{Evaluate}\left[\text{Table}[\text{Normal}[\text{Series}[\text{Exp}[x] * \text{Cos}[x], \{x, 0, n\}]], \{n, 6\}]\right], \{x, -\text{Pi}, \text{Pi}\}\right]$$

$$\text{P2} = \text{Plot}[\text{Exp}[x] * \text{Cos}[x], \{x, -\text{Pi}, \text{Pi}\}, \text{PlotStyle} \to \{\text{Thickness}[0.01]\}]$$

and then the command, Show[P1, P2], we get up to the best 6th degree polynomial approximation. Note that only 4 of these are different since the coefficient of x^2 and the coefficient of x^6 are both 0:

$$\text{Series}[\text{Exp}[x] * \text{Cos}[x], \{x, 0,6\}] = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + O(x^7)$$