Math 1B - Chapter 6 Problems in Anticipation of the Chapter 6 Test

1. Find the volume generated when the region bounded by $0 \le y \le x(2-x)^2$

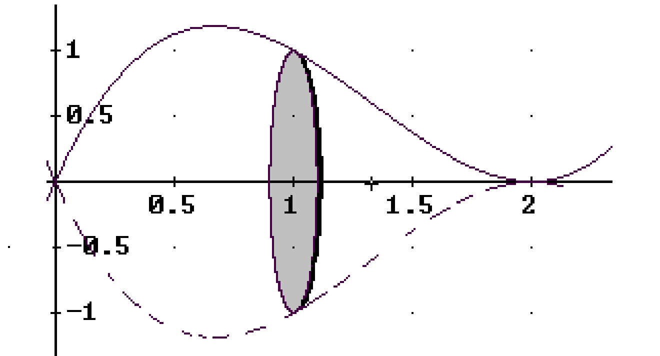
   a. …is revolved about the x-axis.

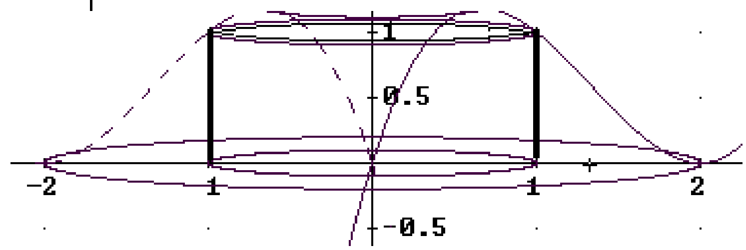
RSP: $V = \pi\int_0^2 \left[x(x-2)^2\right]^2 dx$

$= \pi\int_0^2 x^2(x-2)^4\,dx = \pi\int_{-2}^0 (u+2)^2 u^4 du$

$= \pi\left(\frac{u^7}{7}+\frac{2u^6}{3}+\frac{4u^5}{5}\right)\Bigg|_{-2}^0$

$= \pi\left(\frac{128}{7}-\frac{128}{3}+\frac{128}{5}\right) = \frac{128\pi}{105} \approx 3.83$





   b. …is revolved about the y-axis.

$V = 2\pi\int_0^2 rhdx = 2\pi\int_0^2 x\left(x(2-x)^2\right)dx$

$= 2\pi\int_0^2 4x^2 - 4x^3 + x^4 dx =$

$= 2\pi\left(\frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5\right)\Bigg|_0^2 = \; = 2\pi\left(\frac{32}{3}-16+\frac{32}{5}\right) = 2\pi\left(\frac{160-240+96}{15}\right) = \frac{32\pi}{15} \approx 6.70$

2. Set up an integral for the volume generated when the region bounded by $0 \le y \le x(2-x)$

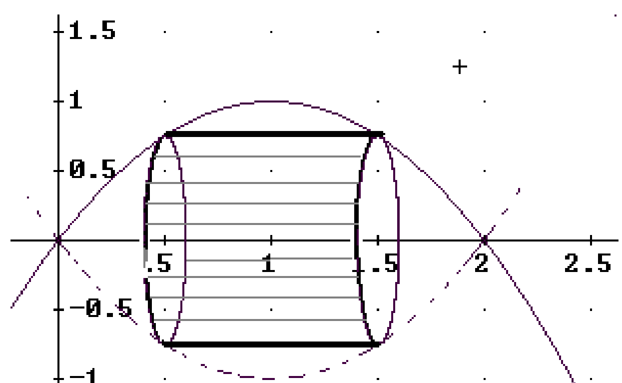
a. …is revolved about the x-axis, using the shell method. *Note: you may need to solve for x.*

ANS: Solving for x we get

$y = x(2-x) = -\left(x^2-2x\right)$

$= -(x-1)^2 + 1 \Leftrightarrow (x-1)^2 = 1-y$

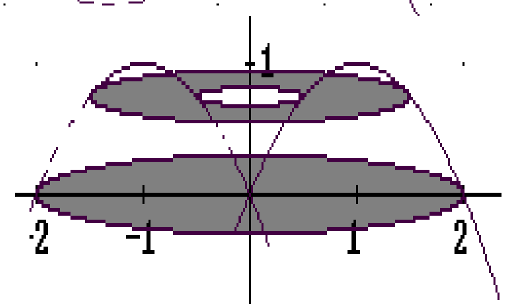
whence $x = 1 \pm \sqrt{1-y}$ so that



$dV = 2\pi rh\,dy = 2\pi y\left[\left(1+\sqrt{1-y}\right)-\left(1-\sqrt{1-y}\right)\right]dy$

$= 4\pi y\sqrt{1-y}\,dy$

so that $V = \int_0^1 4\pi y\sqrt{1-y}\,dy$



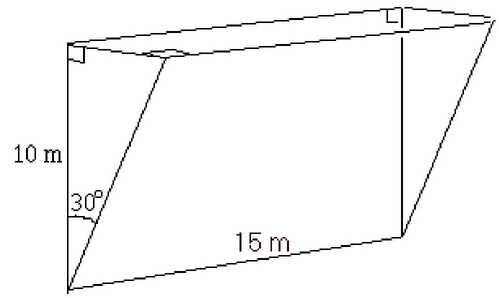
3. …is revolved about the y-axis, using the washer method. *Note: you may need to solve for x.*

ANS: Here $v = \pi\int_0^1\left[\left(1+\sqrt{1-y}\right)^2 - \left(1-\sqrt{1-y}\right)^2\right]dy$

$= 4\pi\int_0^1 \sqrt{1-y}\,dy$

4. Find the volume generated by revolving the area in the first quadrant bounded by $y = x\cos(\pi x)$ and $y = 4x(1-2x)$ about the *y*-axis.

5. Suppose the trough shown below is filled with water. The 10 meter length is vertical. And the front face is tilted at a 30° angle, as shown: Find the minimum total work required to empty the water out through the top of the trough.



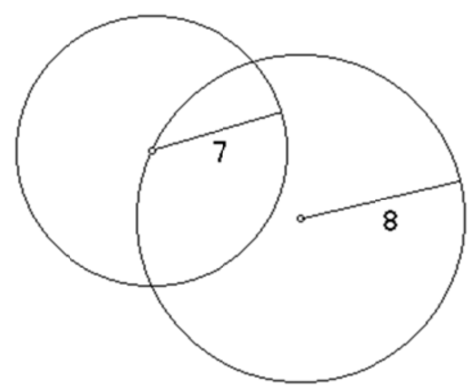
6. Find the volume generated by revolving the area in the first quadrant bounded by $y = 2x\sin(\pi x)$ and $y = 8x^3$ about the *y*-axis.

7. Consider the area of the region bounded by $y = x^2$ and $y = 2-|x|$.
   a. Sketch a graph illustrating the region.
   b. By integrating over *x* and using symmetry, as appropriate.
   c. By integrating over *y* and splitting the region into 2 pieces, as appropriate.

8. Consider the area of the region bounded by $y = (x-1)^2$ and $y = 1+\sin\left(\frac{\pi x}{2}\right)$.
   a. Sketch a graph illustrating the region.
   b. Compute the volume of revolution generated by revolving the region about the *x*-axis. Use either the shell method or the washer method, whichever seems easier.
   c. Compute the volume of revolution generated by revolving the region about the *y*-axis. Use either the shell method or the washer method, whichever seems easier.

9. A hole of radius *r* is drilled through the center a sphere of radius $R > r$. Draw a diagram, introduce a coordinate system, set up an integral and simplify it to find a formula for the volume of the remaining portion of the sphere.

10. Consider the region bounded by two circles with radii 7 and 8, where the circumference of the larger circle passes through the center of the smaller circle, as shown:
   a. Introduce an appropriate coordinate system and sketch a graph showing how these circles are situated in your coordinate system.
   b. Write equations for each circle relative to your coordinate system. These equations may be parametric, if you like.
   c. Set up an integral to compute the area of this region in terms of one of your coordinate variables or the coordinate parameter. Don't simplify the integral.
   d. Set up an integral to compute the volume generated by revolving this area around the line through the two circle centers using the 'shell' method.
   e. Set up an integral to compute the volume generated by revolving this area around the line through the two circle centers using the 'washer' method.

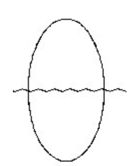


11. The area of the plane bounded between $f(x) = x^2 - 4x + 5$ and $g(x) = 2x - 3$ is revolved around the x-axis. Find the volume of revolution.
    a. Sketch a graph of the region bounded and the volume of revolution.
    b. Set up an integral for the volume of revolution using the washer method.
    c. Set up an integral for the volume of revolution using the shell method.
    d. Evaluate one of these integrals. Use a calculator if necessary.

12. Consider the region $\mathscr{R}$ in the first quadrant bounded by the y-axis and the curves $y = 2\cos x$ and $y = \sin x$.
    Set up (but **do not evaluate**) integrals to compute the following:
    a. The area of $\mathscr{R}$ by integrating over x.
    b. The area of $\mathscr{R}$ by integrating over y.
    c. The volume of the region generated by rotating $\mathscr{R}$ about the y-axis.
    d. The volume of the region generated by rotating $\mathscr{R}$ about the x-axis.
    e. The surface area of the volume generated by revolving $\mathscr{R}$ about the line $x = \arctan(2)$.

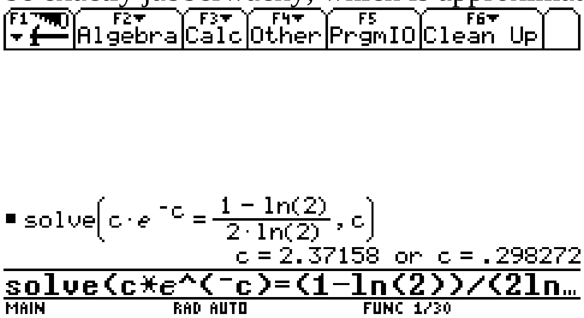
13. Suppose a pyramid with a square base of area 225 square meters and a height of 160 meters is filled with water. Find the work required to pump all the water out of the top of a pyramid.
    SOLN: Whoops: I guess the true dimension of the great pyramid of Egypt is more like the length of an edge of the square base is 225 meters. But we solve the problems we are given, so here it goes:
    The horizontal cross-sections are squares whose side lengths varies linearly from 15 to 0 as the height y ranges from 0 to 160. Thus the cross sectional area at height y is $A(y) = \left(15 - \frac{15y}{160}\right)^2$ - or – if you think of x as the distance from the top, then the cross-sectional area at x is $A(x) = \left(\frac{15x}{160}\right)^2 = \frac{9x^2}{1024}$. This is maybe easier to work with. Then an element of work is $dW = (\text{dist})\,dF = x(9810)\,dV = 9810\left(\frac{9x^3}{1024}\right)dx$. Integrating gives the total amount of work:
    $$\int_0^{160} 9810\left(\frac{9x^3}{1024}\right)dx = \frac{4905(9)}{2048}x^4\Big|_0^{160} = \frac{5^5 \cdot 3^2 \cdot 109 \cdot 2^{20}}{2^{11}} = 3^2 \cdot 109 \cdot 2^4 \cdot 10^5 = 109 \cdot 144 \cdot 10^5 = 1569600000\text{ J}$$

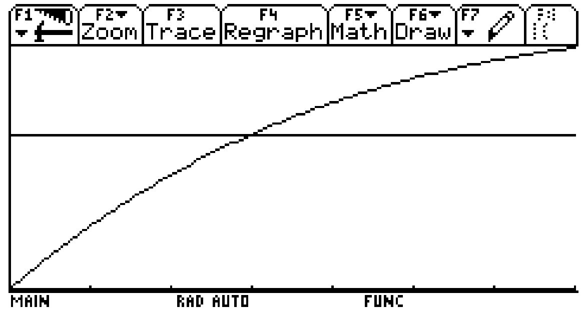
14. Suppose an elliptical thin metal shell with height = 4 meters and horizontal circular cross-sections of radius = 2 meters and is half submerged in water as shown at right.
    What is the weight of the shell? Explain your reasoning.



15. Consider the function $f(x) = x \cdot e^{-x}$ on the interval [0,ln2]. Find the average value of f on the interval.
    SOLN: $f_{\text{AVG}} = \frac{1}{\ln 2}\int_0^{\ln 2} x \cdot e^{-x}dx = \frac{1}{\ln 2}\left(-xe^{-x}\Big|_0^{\ln 2} + \int_0^{\ln 2} e^{-x}dx\right) = \frac{1}{\ln 2}\left(-\frac{\ln 2}{2} - \left(\frac{1}{2} - 1\right)\right) = \frac{1 - \ln 2}{2\ln 2} \approx 0.2213$
    a. Why does this function satisfy the condition of the Mean Value Theorem for Integrals?
       SOLN: $f(x) = x \cdot e^{-x}$ is continuous on the interval [0,ln2].
    b. What equation would you solve to find the number whose existence is guaranteed by the Mean Value Theorem for Integrals for this function on that interval? Can you solve it exactly?
       SOLN: $c \cdot e^{-c} = \frac{1 - \ln 2}{2\ln 2}$ which doesn't a have a nice closed form solution, so it hard to solve it "exactly." We could give the solution a name, like, say, "jabberwacky" and then the solution would

be exactly jabberwacky, which is approximately 0.298272, as the screenshots below show.





16. Consider the region $\mathscr{R}$ in the first quadrant bounded by the y-axis and the curves $y = 2\cos x$ and $y = \sin x$.
Set up (but **do not evaluate**) integrals to compute the following:

a. The area of $\mathscr{R}$ by integrating over x.

SOLN: $\int_0^{\tan^{-1}2} (2\cos x - \sin x)\,dx$

b. The area of $\mathscr{R}$ by integrating over y.

SOLN: $\int_0^{2/\sqrt{5}} \sin^{-1} y\,dy + \int_{2/\sqrt{5}}^{2} \cos^{-1}\frac{y}{2}\,dy$

c. The volume of the region generated by rotating $\mathscr{R}$ about the y-axis.

SOLN: Using shells: $2\pi \int_0^{\tan^{-1}2} x(2\cos x - \sin x)\,dx$

With washers: $\pi\left[\int_0^{2/\sqrt{5}} \left(\sin^{-1} y\right)^2 dy + \int_{2/\sqrt{5}}^{2} \left(\cos^{-1}\frac{y}{2}\right)^2 dy\right]$

d. The volume of the region generated by rotating $\mathscr{R}$ about the x-axis.

SOLN: Using washers: $\pi \int_0^{\tan^{-1}2} (2\cos x)^2 - (\sin x)^2\,dx$

With shells: $2\pi\left[\int_0^{2/\sqrt{5}} y\left(\sin^{-1} y\right) dy + \int_{2/\sqrt{5}}^{2} y\left(\cos^{-1}\frac{y}{2}\right) dy\right]$