Math 1B Chapter 11 Test Solutions 12/9/14

1. (14 points) Find a value of r so that $\sum_{n=0}^{\infty} r^n = 42$

   Solution: $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} = 42 \Leftrightarrow \boxed{r = \frac{41}{42}}$.

2. (18 points) Consider the series $S = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(n)!}$.

   (a) Explain why $\lim_{n\to\infty} \frac{2^n}{(n)!} = 0$

   SOLN: Approach #1: Let $\frac{2^n}{n!}$ so that $y^{1/n} = \frac{2}{(n!)^{1/n}} = \sqrt[n]{\frac{2}{n} \cdot \frac{2}{n-1} \cdots \frac{2}{1}}$.
   It is known that the geometric mean of n positive numbers is always less than the arithmetic mean of the numbers, so $y^{1/n} \le \frac{\frac{2}{n} + \frac{2}{n-1} + \cdots + \frac{2}{1}}{n} = \frac{2}{n} \sum_{i=1}^{n} \frac{1}{i} < \frac{2}{n}\left(1 + \int_1^n \frac{dx}{x}\right) = \frac{2}{n}(1+\ln(n)) \to 0$ as $n \to \infty$ If $y^{1/n} \to 0$ then $y \to 0$.
   Approach #2 (simpler) The series is absolutely convergent by direct comparison: $\sum_{n=0} \frac{2^n}{n!} < 3 + \sum_{n=2}^{\infty} \frac{4}{(n-1)n} < 3 + \sum_{n=2}^{\infty} \frac{4}{(n-1)^2} = 3 + \sum_{n=1}^{\infty} \frac{4}{n^2} = 3 + \frac{2\pi^2}{3}$ converges so the n^{th} term test (a necessary condition for convergence) must be satisfied, which means $\lim_{n\to\infty} \frac{2^n}{(n)!} = 0$. Note that we went a little further than needed here, since our comparable series is a p-series with $p = 2 > 1$, we didn't have to say what it converges to, just that it converges, but this is the famous Basel problem! - the thrill of usage is irresistable.
   Approach #3 (simplest) We can write $\frac{2^n}{n!} = \frac{2}{1}\frac{2}{2}\frac{2}{3}\frac{2}{4}\frac{2}{5}\cdots\frac{2}{n}$. The product of the first four terms is $\frac{2}{3}$. From the fifth term on, each new factor is less than $\frac{1}{2}$. For $n > 4$, $\frac{2^n}{n!} < \frac{2}{3}\left(\frac{1}{2}\right)^{n-4} \to 0$ as $n \to \infty$.

   (b) Explain why $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$ is convergent and use the alternating series error bound to find a value of N so that $\sum_{n=0}^{N} \frac{(-1)^n 2^n}{(n)!}$ approximates $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(n)!}$ to within 0.1

   SOLN: The series converges absolutely, so the alternating series is within $|a_{n+1}|$ of n^{th} partial sum. We seek the smallest n such that $\frac{2^n}{n!} < \frac{1}{20}$ Tabulating, we find

n	0	1	2	3	4	5	6
$\frac{2^n}{n!}$	1	2	2	$\frac{4}{3}$	$\frac{2}{3}$	$\frac{4}{15}$	$\frac{4}{45}$

   Substituting $x = -2$ in the Maclaurin series for e^x we have $e^{-2} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} \approx 0.1353352832366126918939995$

N	0	1	2	3	4	5	6
$\sum_{n=0}^{N} \frac{(-1)^n 2^n}{n!}$	1	-1	1	$\frac{-1}{3}$	$\frac{1}{3}$	$\frac{1}{15} = 0.0\overline{6}$	$\frac{7}{45} = 0.1\overline{5}$

   The table shows we're within 0.1 by $N = 5$.

3. (16 points) Show that the series $\sum_{n=100}^{\infty} \frac{1}{x^{1.1}+1}$ is convergent by using a comparison test and the integral test. Be sure to state why the conditions for using the integral test are met.
   SOLN: $\sum_{n=100}^{\infty} \frac{1}{x^{1.1}+1} \le \sum_{n=100}^{\infty} \frac{1}{x^{1.1}}$. Since the terms of this sequence positive and decreasing, we can test for convergence with the integral test: $\int_1^{\infty} \frac{dx}{x^{1.1}} = \lim_{b\to\infty} -10x^{-0.1}\Big|_1^b = 10$. Since the integral converges, the series converges.

4. (24 points) Find a Maclaurin series for each of the following:

(a) $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!}$.

We can use Mathematica to compute and plot the Taylor polynomials as follows. Start by defining the function we want to approximate:

```
f[x_] := (1+Cos[2*x])/2
```

Then, while Mathematica has a built-in command to compute Taylor polynomials called `Series`, the result is in a special format. To get Taylor polynomials in a more usable form, define a new function, `P`, which takes the order of the polynomial `n`, the name of the function, `f`, the point about which we are forming the Taylor polynomial, `a`, and the variable name `x`:

```
P[n_, f_, a_, x_] := Normal[Series[f[t], {t, a, n}]] /. t -> x
```

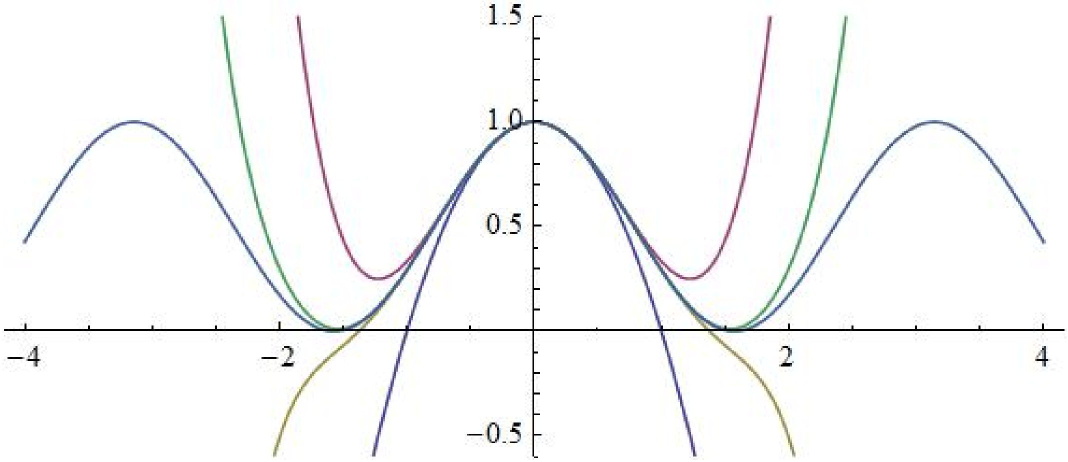
Now this command:

```
P[12, f, 0, x]
```

will produce $\frac{2x^{12}}{467775} - \frac{2x^{10}}{14175} + \frac{x^8}{315} - \frac{2x^6}{45} + \frac{x^4}{3} - x^2 + 1$ and then

```
Plot[{P[2, f, 0, x], P[4, f, 0, x], P[6, f, 0, x], P[8, f, 0, x],
  f[x]}, {x, -4, 4}, PlotRange -> {-0.6, 1.5}]
```

will give us

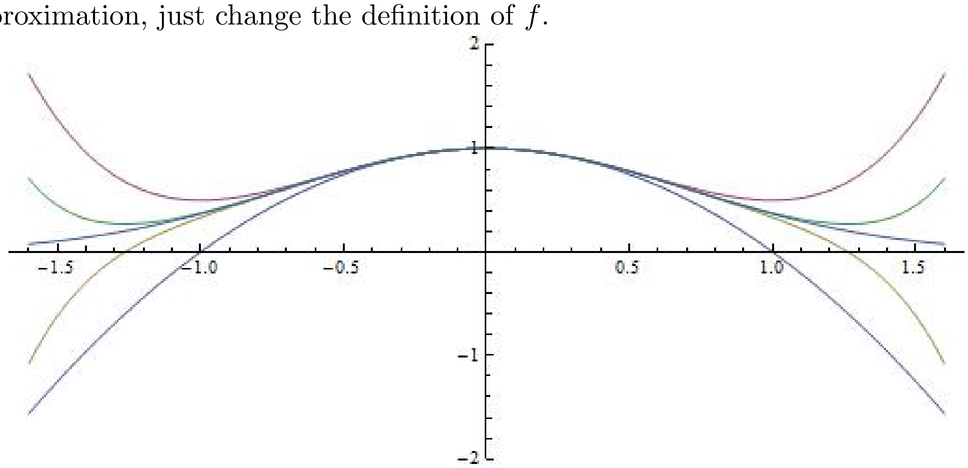


You can also take the more round-about approach and compute coefficients from the formula, keeping in mind that this is an even function and so all the odd coefficients will be 0:

$c_0 = f(0) = 1$ Now $f'(x) = -2\cos(x)\sin(x) = -\sin(2x)$ allows us to easily compute, for $n > 0$ even,

$c_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^n 2^{2n-1}}{(2n)!} \Rightarrow \cos^2(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} x^{2n}$

(b) $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$. We can again use Mathematica to look at the plots of the polynomial approximation, just change the definition of f.

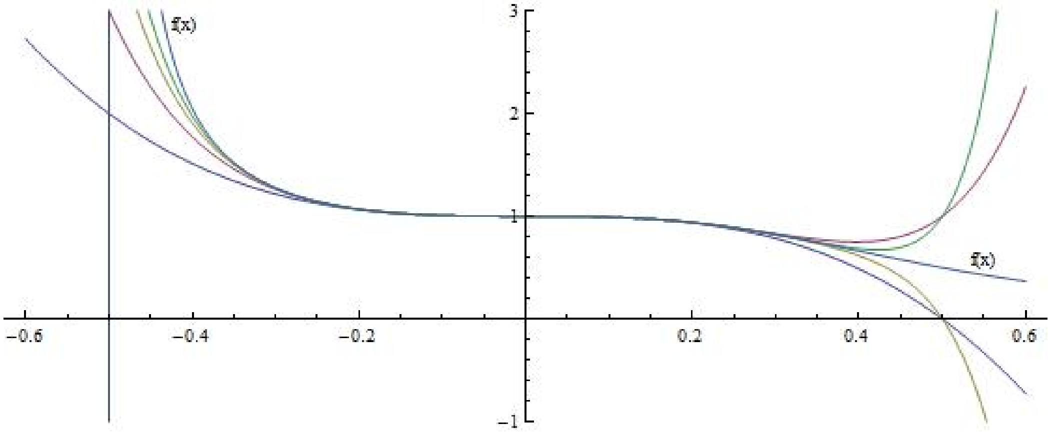


(c) $\frac{1}{1+(2x)^3} = \frac{1}{1-(-2x)^3} = \sum_{n=0}^{\infty} (-8x^3)^n = \sum_{n=0}^{\infty} (-1)^n 8^n x^{3n}$

The radius of convergence can be found by using the ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} 8\left|x\right|^3 < 1 \Leftrightarrow \frac{-1}{2} < x < \frac{1}{2}$ and the radius is $\frac{1}{2}$. The plot of the polynomials generated by

```
Plot[{P[3, f, 0, x], P[6, f, 0, x], P[9, f, 0, x], P[12, f, 0, x],
  f[x]}, {x, -0.6, 0.6}, PlotRange -> {-1, 3}]
```

looks like this:



5. (12 points) Estimate the error in each approximation.

(a) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n} \approx \sum_{n=1}^{10}\frac{(-1)^n}{n}$. SOLN: Since this is an alternating series with terms of diminishing magnitude, the error in approximation is less than the value of the first neglected term: $\frac{1}{11} \approx 0.\overline{09}$. This alternating harmonic series which converges to $-\ln 2 \approx -0.6931471805599453094172321214581 8$. Tabulating some partial sums,

n	6	7	8	9	10	11
S_n	$-0.61\overline{6}$	$-0.7\overline{5952380}$	$-0.6\overline{34523809}$	$-0.745\overline{634920}$	$-0.645\overline{634920}$	$-0.\overline{736544011}$
Error	0.07648	0.06638	0.05862	0.05249	0.04751	0.04340

we see that, while the error at the 10th term is less than half of a tenth, the rounding error locks in the correct digit only after the 10th term.

(b) $\sum_{n=1}^{\infty}\frac{1}{n^2+2n} \approx \sum_{n=1}^{10}\frac{1}{n^2+2n}$. SOLN: Since the terms in this series are positive and decreasing we can apply the integral test remainder formulas to the error $R_n = \left|S - S_n\right|$:

$$\int_{11}^{\infty}\frac{1}{x^2+2x}\,dx < R_n < \int_{10}^{\infty}\frac{1}{x^2+2x}\,dx$$

The integral can be done either by trig substitution or partial fractions. Since these are both methods worth reviewing, let's start with the harder way first: $\int_a^{\infty}\frac{1}{(x+1)^2-1}\,dx$ suggests the trig substituion

$x+1 = \sec\theta, dx = \sec\theta\tan\theta\,d\theta$ giving the integral $\int_{\alpha}^{\pi/2}\frac{\sec\theta\tan\theta}{\tan^2\theta}\,d\theta$ where $\alpha = \arctan\frac{1}{a+1}$. The integral simplifies to $\int_{\alpha}^{\frac{\pi}{2}}\csc(x)\,dx = -\ln\left|\csc\theta+\cot\theta\right|_{\alpha}^{\pi/2} = \ln\left|\csc\alpha+\cot\alpha\right| = \ln\left(\frac{12}{\sqrt{143}}+\frac{1}{\sqrt{143}}\right) = \ln\left(\frac{13}{\sqrt{143}}\right)$ if $a = 11$ and $\ln\left(\frac{6}{\sqrt{30}}\right)$ if $a = 10$.

With partial fractions, $\int_a^{\infty}\frac{1}{x^2+2x}\,dx = \frac{1}{2}\int_a^{\infty}\frac{1}{x}-\frac{1}{x+2}\,dx = \frac{1}{2}(\ln(x)-\ln(x+2))\Big|_a^{\infty} = \frac{1}{2}\ln\frac{x}{x+2}\Big|_a^{\infty}$
$= \ln\sqrt{\frac{a+2}{a}}$. If $a = 11$ this is $\ln\sqrt{\frac{13}{11}}$ and $a = 10$ gives $\ln\sqrt{\frac{6}{5}}$. It's easy to check that these are equivalent to the trig-based values. Thus $\ln\sqrt{\frac{13}{11}} \approx 0.08353 < R_n < 0.09116 \approx \ln\sqrt{\frac{6}{5}}$

6. (16 points) Find the first three terms of the Maclaurin series for $\sqrt{9-x}$ and use these to approximate $\sqrt{8}$. How accurate is your approximation?
$\sqrt{9-x} = 3\sqrt{1-x/9} = 3\sum_{n=0}^{\infty}\binom{1/2}{n}\left(\frac{-x}{9}\right)^n \approx 3(1-\frac{x}{18}-\frac{1}{648}x^2)$, $x=1$ for $\sqrt{8} \approx 3(1-0.0\overline{5}-0.00154) \approx 2.8287\overline{037}$

7. Use series to approximate $\int_0^1 e^{-x^2}\,dx$ to the nearest hundredth.

SOLN: $\int_0^1 e^{-x^2}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty}\int_0^1 \frac{(-1)^n x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}\Big|_0^1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}$
$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} + \cdots$. Since the series is alternating with the terms decreasing in magnitude, the error is less than the first neglected term. The true value of the series is $\frac{\sqrt{\pi}}{2}\texttt{erf}(1) \approx 0.746824132812427$

n	2	3	4	5
S_n	$0.7\overline{6}$	$0.7\overline{428571}$	0.747487	0.746729
R_n	0.0198	−0.0040	0.00066	−0.000095