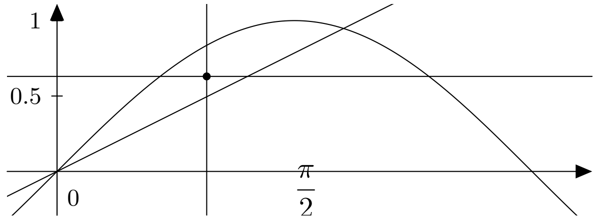
Math 1B Chapters 8 and 10 Test Solutions

Note: A variety of CAS are used here, including Geogebra, Mathematica and Sage, which uses Macysma

1. A thin sheet of metal is shaped like the region in the first quadrant between $y = \sin x$ and $y = \frac{x}{2}$.



(a) Graph the region.

(b) Find the area of the region.

To determine the point of intersection we solve $\sin(x) = \frac{x}{2}$. Since this equation doesn't have a closed-form solution, we need to find an approximation we can use. Newton's method comes to mind. The equation of the tangent line at the point $(x_n, f(x_n))$ is given by the equation

$$y - f(x_n) = f'(x_n)(x - x_n), \tag{1}$$

where $f'(x_n)$ is the first derivative of f evaluated at x_n.[1] To find the x-intercept of the tangent line, substitute $y = 0$ and solve for x.

$$\begin{aligned} 0 - f(x_n) &= f'(x_n)(x - x_n) \\ x - x_n &= -\frac{f(x_n)}{f'(x_n)} \\ x &= x_n - \frac{f(x_n)}{f'(x_n)} \end{aligned}$$

Thus, the next term in the sequence of intercepts that are converging to a root of f is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2}$$

This defines *Newton's Method* for finding a root of a given function.

In this case we get

$$x_{n+1} = x_n - \frac{\sin(x_n) - x_n/2}{\cos(x_n) - 1/2}. \tag{3}$$

n	0	1	2	3	4	5
x_n	1.6	1.98	1.899	1.89550	1.89549426706	1.89549426703398094

This illustrates the usual accurate digit doubling behavior of Newton's method, so at the end we have 18 correct digits. Let $\alpha = 1.89549426703398094$ ...or the exact convergent, which is unknown. Then the area is

$$A = \int_0^\alpha \sin(x) - \frac{x}{2}\,dx = -\cos(x) - \frac{x^2}{4}\Big|_0^\alpha = 1 - \cos(\alpha) - \alpha^2/4 \approx 0.420797895052946624$$

(c) Find the x-coordinate of the center of mass of the region.

$$\overline{x} = \frac{1}{A}\int_0^\alpha x\sin(x) - \frac{x^2}{2}\,dx = \boxed{\begin{array}{c|c} u = x & dv = \sin(x)dx \\ \hline du = dx & v = -\cos(x) \end{array}} = \frac{1}{A}\left(-x\cos(x) + \sin(x) - \frac{x^3}{6}\right)\Big|_0^\alpha$$

$$= \frac{1}{A}\left(-\alpha\cos(\alpha) + \sin(\alpha) - \frac{\alpha^3}{6}\right) \approx 0.991923709756319716$$

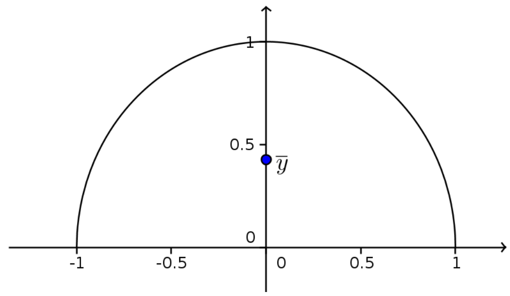
(d) Find the y-coordinate of the center of mass of the region.

$$= \frac{1}{2A}\int_0^\alpha \left(\sin^2 x - x^2/4\right)\,dx = \frac{1}{2A}\left(\frac{\alpha}{2} - \frac{\sin(2\alpha)}{4} - \frac{\alpha^3}{12}\right) \approx 0.63141586040866049$$

[1] Calculus students will recall that the first derivative evaluated at x_n provides the slope of the tangent line to the graph of f at the point $(x_n, f(x_n))$.

2. An art student was given a circular metal disk 2 ft in diameter and told to drill a hole in it so that when the disk is cut in half and the piece with the hole is placed atop a spike stuck in the hole, it will balance. Not knowing about integrals, the artist drilled a hole at a point half-way between the center and the edge.

(a) Where *should* the student have drilled the hole?
SOLN: Suppose the semidisk is oriented in the xy–plane as shown at right. By symmetry, $\overline{x} = 0$. There is a choice of integrals for the moment about the x–axis: $M_x = \int_0^1 2y\sqrt{1-y^2}\,dy$
$= -\frac{2}{3}(1-y^2)^{3/2}\Big|_0^1 = \frac{16}{3} \Rightarrow \overline{y} = \frac{2/3}{\pi/2} \approx 0.4244$
Alternatively, $\overline{y} = \frac{1}{2A}\int_{-1}^{1}(1-x^2)\,dx = \frac{4}{3\pi}$



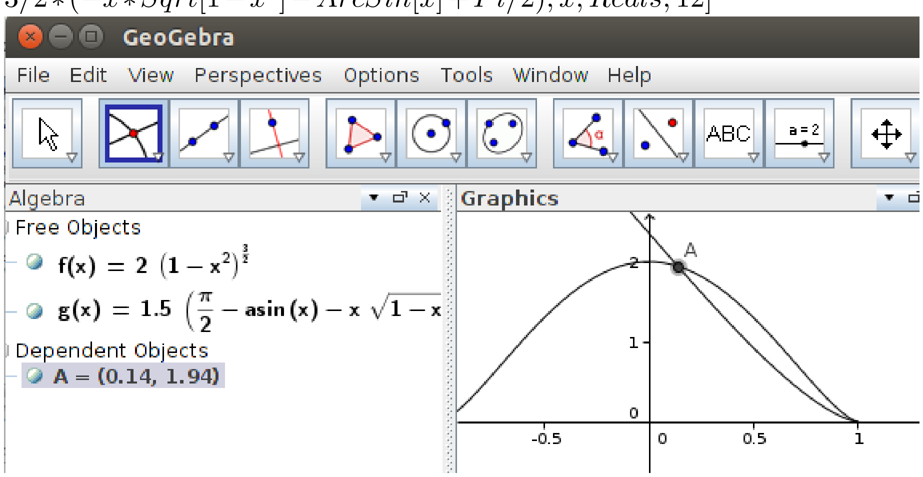
(b) Now that she has made the mistake, she decides that rather than drill a second hole, she will cut the piece with the hole in it in such a way that it will balance on the spike at the point of the hole. Explain clearly how the disk should be cut so that our artist friend can understand.
SOLN: The simplest thing would be to slice along a cord parallel to the diameter cut already made. We want to choose $y = a$ so that $\overline{y} = \frac{1}{2} \Leftrightarrow \frac{1}{A}\int_a^1 2y\sqrt{1-y^2}\,dy = -\frac{2}{3A}(1-y^2)^{3/2}\Big|_a^1 = \frac{2}{3A}\left(1-a^2\right)^{3/2} = \frac{1}{2}$
where $A = \int_a^1 2\sqrt{1-y^2}\,dy$. The area integral can be done by trig substitution $y = \sin\theta$, yielding

$$A = \int_{\arcsin(a)}^{\pi/2} 2\cos^2\theta\,d\theta = \theta + \frac{\sin 2\theta}{2}\Big|_{\arcsin(a)}^{\pi/2} = \frac{\pi}{2} - \arcsin(a) - a\sqrt{1-a^2}$$

Thus the equation becomes $2\left(1-a^2\right)^{3/2} = \frac{3}{2}\left(\frac{\pi}{2} - \arcsin(a) - a\sqrt{1-a^2}\right)$

One way to approximate the solution to this equation is to graph the left side and the right side and to seek the coordinates of intersection. That's what the Geogebra session (at right) shows. In Mathematica, the more precise value of $\{x \to 0.138172703842\}$ is obtained using the `NSolve` command:

$NSolve[2*(1-x^2)^(3/2) == 3/2*(-x*Sqrt[1-x^2] - ArcSin[x] + Pi/2), x, Reals, 12]$



(c) What is the area of the piece of the metal disk that balances at the point where the hole $\left(0, \frac{1}{2}\right)$ was drilled?
SOLN: The trimmed semicircle will have area $\approx \int_{0.138172703842}^{1} 2\sqrt{1-y^2}\,dy \approx 1.29533$

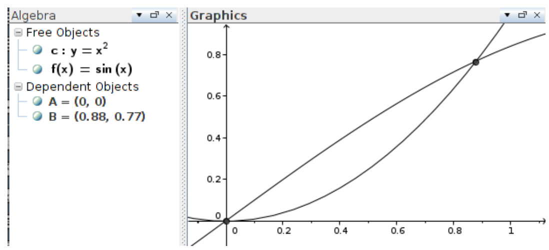
3. In each of the following, find the length of the given arc and use Pappus' theorem to find the volume of the solid obtained by rotating the given region, $\mathscr{R}$

   (a) Arc: $y = \sin x$ from $x = 0$ to $x = \pi$.
   $\mathscr{R}$: One arch of the sine curve above the x–axis.
   SOLN: Arc length $= \int_0^\pi \sqrt{1+\cos^2(x)}\,dx = 2\sqrt{2}E\left(\frac{1}{2}\right) \approx 3.8202$, Where E is the complete integral of the second kind.
   Since the x–coordinate of the region $\mathscr{R}$ is simply, by symmetry $\overline{x} = \dfrac{\pi}{2}$ and the area of the region is $\int_0^\pi \sin(x)\,dx = -\cos(x)\Big|_0^\pi = 2$ the volume of revolution is $2\pi(\dfrac{\pi}{2}(2) = 2\pi$ cubic units.

   (b) Arc: $y = x^2$ from $x = 0$ to $x = 1$.
   SOLN: Arc length $= \int_0^1 \sqrt{1+4x^2}\,dx$ Substitute $x = \frac{1}{2}\tan\theta\,, dx = \frac{1}{2}\sec^2\theta\,d\theta$ and the integral becomes $\frac{1}{2}\int_0^{\arctan 2} \sec^3\theta d\theta = \frac{1}{4}\sec\theta\tan\theta + \ln|\tan\theta+\sec\theta|\Big|_0^{\arctan 2} = \frac{1}{2}\sqrt{5} + \frac{1}{4}\ln|\sqrt{5}+2|$
   $\mathscr{R}$: The region between $y = x^2$ and $y = \sin x$ in the first quadrant.

   SOLN:We can use Geogebra to see the region in question and approximate the coordinates of the point of intersection (at right) but it only gives a two-digit approximation, so I used sage to get a better estimate:

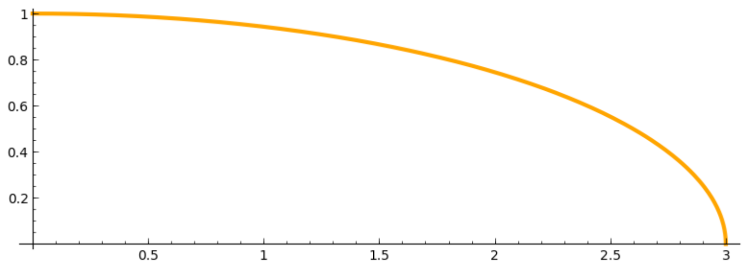
```
sage: find_root(x^2==sin(x),.5,1)
0.8767262153950625
```



   Let $\alpha = 0.8767262153950625$. Then the area is $A \approx \int_0^\alpha \sin(x) - x^2\,dx = 1 - \cos\alpha - \frac{\alpha^3}{3} \approx 0.13569750723060278$
   $M_y \approx \int_0^\alpha x\sin(x) - x^3\,dx = -x*\cos(x) + \sin(x) - \frac{x^4}{4}\Big|_0^\alpha = 0.0601272920687$ $M_x \approx \frac{1}{2}\int_0^\alpha \sin^2(x) - x^4\,dx \approx 0.0444621345032$
   Thus $\overline{x} = \dfrac{0.0601272920687}{0.13569750723060278} \approx 0.443097985334$
   and $\overline{y} = \dfrac{0.0444621345032}{0.13569750723060278} \approx 0.327656236364$
   Thus the volume of revolution about the x–axis is $A \cdot 2\pi\overline{y} \approx 0.279363835937499$
   and the volume of revolution about the y–axis is $A \cdot 2\pi\overline{x} \approx 0.377790925796901$

   (c) Arc: $y = \sqrt{1 - \dfrac{x^2}{9}}$ from $x = 0$ to $x = 3$ (The integral is a special case of an *elliptic integral*).
   SOLN: Arc length $= \int_0^3 \sqrt{1 + \frac{x^2}{9(9-x^2)}}\,dx$ We can use Sage to aid our analysis:

```
sage: var('x');
sage: #Define a 2d curve explicitly y=f(x) and an interval for x.
sage: f=sqrt(1-x^2/9)
sage: #Let's plot C
sage: xmin=0; xmax=3; ymin=0; ymax=1;
sage: C=plot(f,(x,0,3),color='orange',thickness=3)
sage: show(C,aspect_ratio=1)
```



From the graph it's evident the arc length is more than 3.2 and less than 4.

The commands

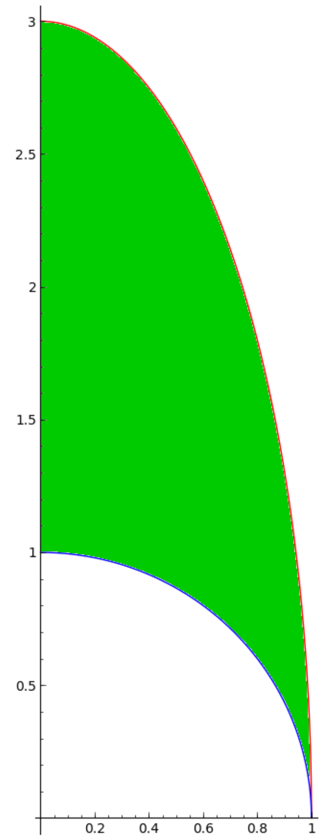
```
df=diff(f,x)
view(df)
```

Produce $-\dfrac{x}{9\sqrt{-\frac{1}{9}x^2+1}}$ which is not the way I would write it, but equivalent. Using Sage to compute the arc length, we write

```
sage: df=diff(f,x)
sage: Lexact=integral(sqrt(1+df^2),(x,0,3))
sage: n(Lexact)
```

This produces $\displaystyle\int_0^3 \sqrt{-\frac{x^2}{9(x^2-9)}+1}\,dx = 3E\left(\frac{8}{9}\right) \approx 3.341223276776997$

$\mathscr{R}$: The region in the first quadrant that lies beneath $y = \sqrt{1-\dfrac{x^2}{9}}$ and outside the unit circle.



The area of a quarter ellipse with semiminor axis = 1 and semimajor axis = 3 is $\dfrac{3\pi}{4}$. Subtracting the area of the quarter unit circle gives the area of the region = $\dfrac{\pi}{2}$

In the diagram at right, the axes are reversed from the usual perspective, so with that we'd have $f(x) = 3\sqrt{1-x^2}$ and $g(x) = \sqrt{1-x^2}$ so that

$M_y = \int_0^1 x(3\sqrt{1-x^2} - \sqrt{1-x^2}\,dx = \int_0^1 2x\sqrt{1-x^2}\,dx = \dfrac{2}{3}$

$\Rightarrow \overline{x} = \dfrac{4}{3\pi} \approx 0.424413181578388.$

$M_x = \frac{1}{2}\int_0^1 (3\sqrt{1-x^2})^2 - (\sqrt{1-x^2})^2\,dx = 4\int_0^1 (1-x^2)\,dx = \dfrac{8}{3}$

Thus $\overline{y} = \dfrac{8/3}{\pi/2} = \dfrac{16}{3\pi} \approx 1.69765272631355$

The volume of revolution about the x–axis is $2\pi\overline{y}A \approx 16.7551608191456$

The volume of revolution about the y–axis is $2\pi\overline{x}A \approx 4.18879020478639$

4. In this problem we will examine the length of the arc of the curve $y = x^n$ on the interval $[0,1]$ for different values of n.

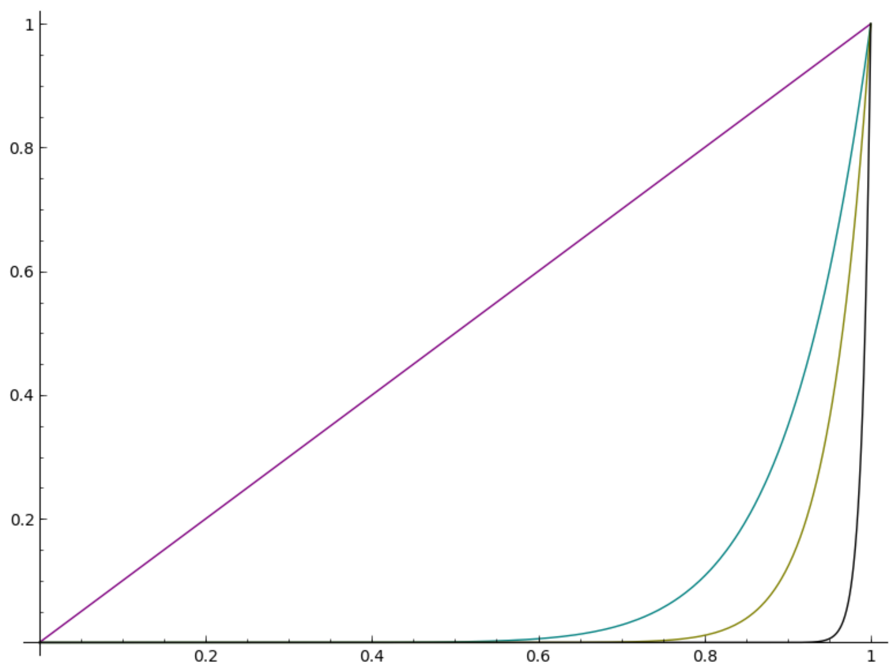
(a) Approximate the length of the arc of the curve $y = x^n$ on the interval $[0, 1]$ for $n = 1, 10, 20$, and 100.
SOLN: Let's try the wonderful Sage for this.

```
sage: f(x)=x
sage: df=diff(f,x)
sage: Lexact=integral(sqrt(1+df^2),(x,0,1))
sage: N(Lexact)
1.41421356237310
sage: f(x)=x^10
sage: df=diff(f,x)
sage: Lexact=integral(sqrt(1+df^2),(x,0,1))
sage: N(Lexact)
1.7544093764948012
sage: f(x)=x^20
sage: df=diff(f,x)
sage: Lexact=integral(sqrt(1+df^2),(x,0,1))
sage: N(Lexact)
1.8421431628157183
sage: f(x)=x^100
sage: df=diff(f,x)
sage: Lexact=integral(sqrt(1+df^2),(x,0,1))
sage: N(Lexact)
1.9516717538525004
```

(b) For the case $n = 1$ explain how you can get the answer very quickly by just looking at the graph.
SOLN: It's the hypotenuse of an isosceles triangle with legs = 1: $\sqrt{2}$

(c) Discuss any pattern or trend you see in the calculations in Part (a).
The lengths are growing and approaching 2 from below.

(d) Plot the graphs of the four curves in Part 1 and use them to help explain what is happening to the arc lengths as n gets larger.

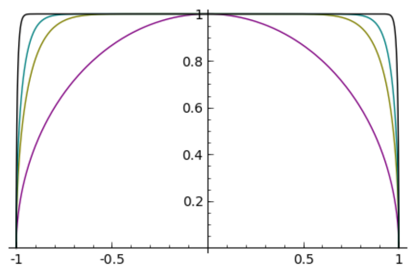


(e) Based on all the above, find $\lim_{n\to\infty} \int_0^1 \sqrt{1 + (n+1)^2 x^{2n}}\, dx$
Intuitively, the limit appears to be approaching 2 from below.

(f) Repeat Parts (a) through (d) using the curve $y = \sqrt{1-x^n}$ on the interval $[-1, 1]$.

```
sage: f(x)=sqrt(1-x^2)
sage: df=diff(f,x)
sage: Lexact=integral(sqrt(1+df^2),(x,-1,1))
sage: N(Lexact)
3.14159265358979
sage: f(x)=sqrt(1-x^10)
sage: df=diff(f,x)
sage: Lexact=integral(sqrt(1+df^2),(x,-1,1))
sage: N(Lexact)
3.5988221155204547
sage: f(x)=sqrt(1-x^20)
sage: df=diff(f,x)
sage: Lexact=integral(sqrt(1+df^2),(x,-1,1)
sage: N(Lexact)
3.737601185558435
sage: f(x)=sqrt(1-x^100)
sage: df=diff(f,x)
sage: Lexact=integral(sqrt(1+df^2),(x,-1,1))
sage: N(Lexact)
3.9161799006525766
```

Numerical and graphical evidence supports the conjecture that $\lim_{n\to\infty} \int_0^1 \sqrt{1 + \frac{x^{2(n-1)}}{4(1-x^{2n})}}\, dx = 4$



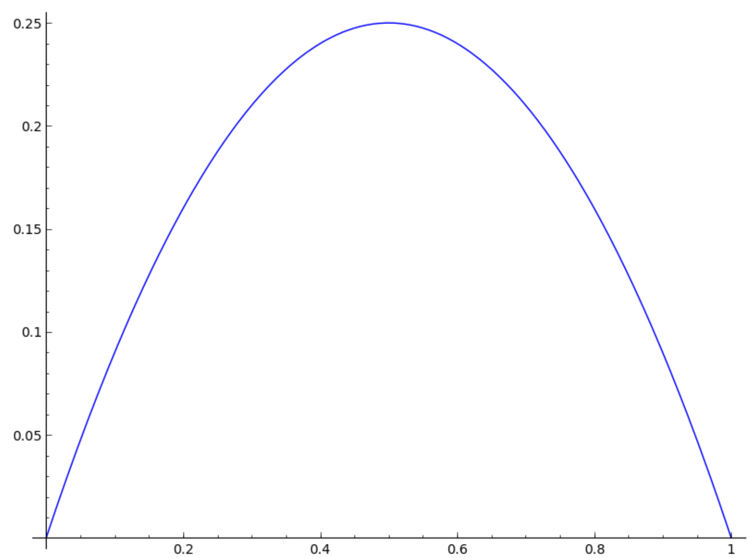
5. We will explore what happens to the ratio of arc length to area on $[0, 1]$ as $a \to \infty$ for four curves that depend on the parameter a. For each of the four functions that follow,
(i) Plot the graph of the function for $a = 1$.
(ii) Find the area bounded by the function and the x–axis on $[0, 1]$.
(iii) With pencil and paper, write down the integral formulas for the arc length on $[0, 1]$ and the area under the curve on $[0, 1]$. Use these to find an integral formula for the limit of the ratio of arc length to area as $a \to \infty$.
(iv) Using your work in Part (iii), find the limit as $a \to \infty$ of the ratio of arc length to area on $[0, 1]$
(v) By looking at the geometry of the graph, can you find a way to predict the limit in Part (iv) without doing the calculations?

(a) $a(x - x^2)$

SOLN: Area $= a\int_0^1 x - x^2\, dx = \frac{ax^2}{2} - \frac{ax^3}{3}\Big|_0^1 = \frac{a}{6}$

Arc length: $L = \int_0^1 \sqrt{1 + a^2(1+2x)^2}\, dx$ Substitute $u = a(1-2x)$ for $L = \frac{1}{2a}\int_{-a}^{a} \sqrt{1+u^2}\, du = \frac{1}{a}\int_0^{\arctan a} \sec^3\theta\, d\theta = \frac{\sqrt{a^2+1}}{2} - \frac{1}{2a}\ln|a + \sqrt{a^2+1}| = \frac{\sqrt{a^2+1}}{2} - \frac{\operatorname{arcsinh}(a)}{2a}$

$\lim_{a\to\infty} \dfrac{\frac{\sqrt{a^2+1}}{2} - \frac{\operatorname{arcsinh}(a)}{2a}}{a/6} = 3$

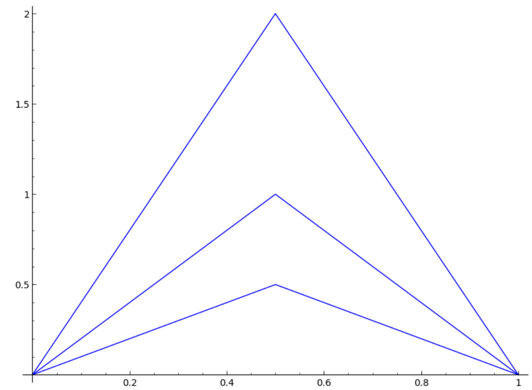


(b) $a\left(\frac{1}{2} - \left|x - \frac{1}{2}\right|\right)$

Area= $\dfrac{a}{4}$

Arc length $L = \dfrac{\sqrt{1+a^2}}{2}$

$\lim_{a\to\infty} \dfrac{\sqrt{1+a^2}/2}{a/4} = 2$



(c) $a\sin(\pi x)$

Area $= a\int_0^1 \sin(\pi x)\,dx = \dfrac{2a}{\pi}$

Arc length $L = \int_0^1 \sqrt{1+a^2\pi^2\cos^2(\pi x)}\,dx.$

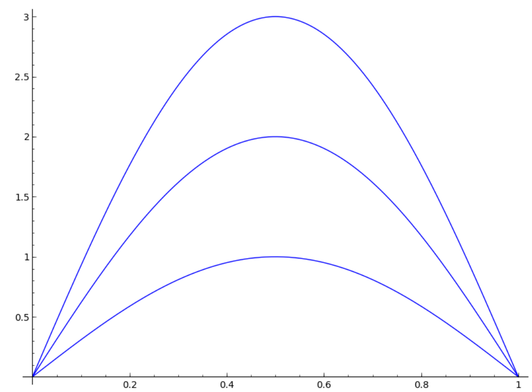
If $a=1$, $L = 2\sqrt{1+\pi^2}E\left(\dfrac{\pi^2}{1+\pi^2}\right)\pi \approx 2.30489$

and the ratio is ≈ 3.62051

To find the limit, a table of values may be helpful:

a	1	10	100
ratio	3.62051	3.150080	3.1428

It becomes evident that this is approaching π.



(d) a times a semicircle of radius 1.

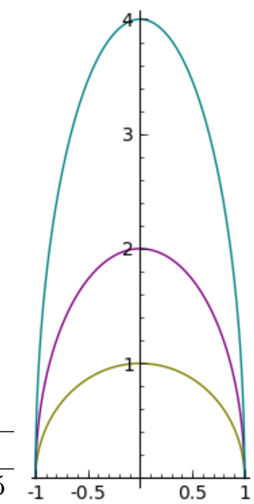
From the well-known formula for an ellipse, Area $= a\pi/2$

Arc length for $a=1$ is π. For $a=2$,

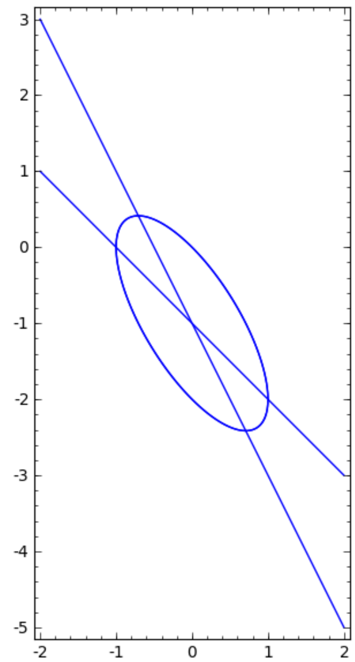
$L = 2\int_0^1 \sqrt{1+\dfrac{4x^2}{1-x^2}}\,dx = 2E(-3) \approx 4.84422407278316$

```
sage: f(x)=4*sqrt(1-x^2)
sage: df=diff(f,x)
sage: Lexact=2*integral(sqrt(1+df^2),(x,0,1))
sage: n(Lexact)
8.57842170017548
```

a	1	4	1024	100000	10000000	1000000000
ratio	2	1.4	1.2732	1.27323952	1.2732395328014	1.2732395328015



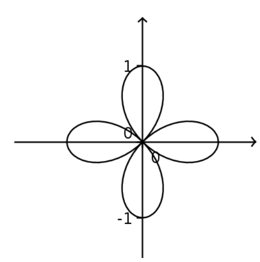
6. Consider a flat metal plate to be placed vertically under water with its top 2 meters below the surface of the water. Determine a shape for the plate so that if the plate is divided into any number of horizontal strips of equal height, the hydrostatic force on each strip is the same.
SOLN: Embarrassing to say how much I struggled with this. When I hit upon the answer is was *really* obvious in hindsight.
Let $f(x) = k/x$ where $k > 0$, then (we can ignore the common factors for force density of the fluid since they're on both sides of the equation) we have $\int_a^{a+h} xf(x)\,dx = kx\Big|_a^{a+h} = k(a+h-a) = kh$ is independent of a, so that does it. Darn the torpedoes!



7. Find the centroid of the region enclosed by the ellipse $x^2+(x+y+1)^2=1$. Note that it's tilted...
SOLN: Rather than rotating axes by $\theta = \frac{1}{2}\arctan(2)$, noting the coordinates of the center and then rotating them back (which proves to be rather laborious), look at the line through the points where the tangent lines are horizontal and the line through the points where the tangent lines are vertical - the intersection of these lines will be the center of the ellipse.
Differentiating implicitly with respect to x we get
$2x+2(x+y+1)\left(1+\dfrac{dy}{dx}\right)=0.$
Setting $\dfrac{dy}{dx}=0$ yields the relation $y=-2x-1$. Similarly, differentiating with respect to y we get $2x\dfrac{dx}{dy}+2(x+y+1)\left(\dfrac{dx}{dy}+1\right)=0$. Setting $\dfrac{dx}{dy}=0$ yields the second relation $y=-x-1$. Solving these two equations simultaneously gives the center of mass at $(\overline{x},\overline{y})=(0,-1)$ as is evident in the graph shown. The graph was done in Sage with

```
sage: p1=implicit_plot(x^2+(x+y+1)^2==1,(x,-2,2),(y,-2.5,1.5))
sage: p2=plot(-1-2*x,(x,-2,2))
sage: p3=plot(-x-1,(x,-2,2))
sage: show(p1+p2+p3)
```

8. Find a formula for the area of the surface generated by rotating the polar curve $r=f(\theta), a\le\theta\le b$ (where f' is continuous and $0\le a<b\le\pi$), about the line $\theta=\pi/2$. Apply this to $r=\cos(2\theta)$.
SOLN: $A=\int dA=\int 2\pi R\,ds=\int 2\pi x\sqrt{\left(\dfrac{dx}{d\theta}\right)^2+\left(\dfrac{dy}{d\theta}\right)^2}\,d\theta = 2\pi\int r\cos\theta\sqrt{\left(\dfrac{d(x\cos\theta)}{d\theta}\right)^2+\left(\dfrac{d(r\sin\theta)}{d\theta}\right)^2}\,d\theta =$
(after expanding and applying Pythagoras' theorem twice) $=2\pi\int_{-\pi/2}^{\pi/2} r\cos\theta\sqrt{\left(\dfrac{dr}{d\theta}\right)^2+r^2}\,d\theta$
With $r=\cos(2\theta)$, this becomes $A=2\pi\int_{-\pi/2}^{\pi/2}\cos(2\theta)\cos\theta\sqrt{4\sin^2 2\theta+\cos^2 2\theta}\,d\theta$
Since the integrand is an even function,
$A=4\pi\int_0^{\pi/2}\cos(2\theta)\cos\theta\sqrt{4\sin^2 2\theta+\cos^2 2\theta}\,d\theta$
Using the product to sum identity and the Pythagorean identity, we get
$A=2\pi\int_0^{\pi/2}(\cos 3\theta+\cos\theta)\sqrt{1+3\sin^2 2\theta}\,d\theta\approx$
5.327462965643178593830910538561289160469444825704024488003896...
This seems like it's in the ball park when you look at the figure being rotated:



9. Find the arclength and area enclosed by the parametric equations for $a=1$ and $a=2$

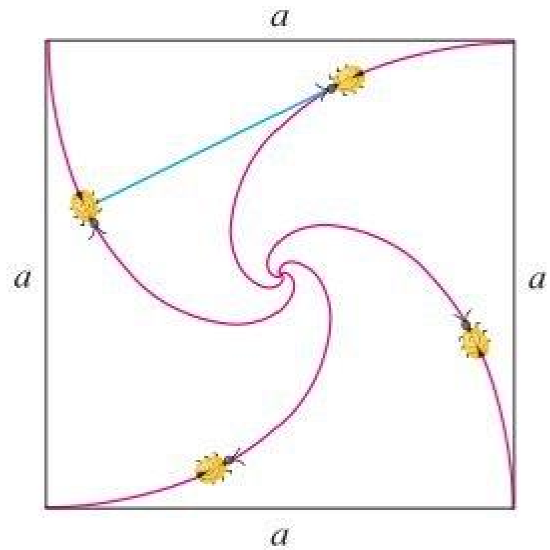
$$x=2a\cos t-a\cos(2t)$$
$$y=2a\sin t-a\sin(2t)$$

SOLN: Area $A=\int y\,dx=2a^2\int_\pi^0(2\sin t-\sin(2t))(2\sin(2t)-2\sin t)\,dt=8a^2\int_0^\pi(2\sin t-\sin(2t))^2\,dt$
$=4a^2\int_0^\pi 2\sin^2 t-3\sin t\sin 2t+\sin^2 2t\,dt=4a^2\int_0^\pi 1-\cos 2t-\frac{3}{2}(\cos t-\cos 3t)+\dfrac{1-\cos 4t}{2}\,dt=$

$= 4a^2\left[t - \frac{1}{2}\sin 2t + \frac{3}{2}\sin t + \frac{1}{2}\sin 3t + \frac{t}{2} - \frac{\sin 4t}{8}\right]_0^{\pi t} = 4a\left(\frac{3\pi}{2}\right) = 6\pi a^2$

Length $L = 2\int\limits_0^{\pi}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ = (after applying Pythagoras' identity twice and the addition identity for cosine) $= 2\int\limits_0^{\pi}\sqrt{8a^2(1-\cos(2t-t))}\,dt = 16a$ (trivial details omitted for brevity.)

10. Four bugs are placed at the four corners of a square with side length a . The bugs crawl counter-clockwise at the same speed and each bug crawls directly toward the next bug at all times. They approach the center of the square along spiral paths.



(a) Find the polar equation of a bugs path assuming the pole is at the center of the square. (Use the fact that the line joining one bug to the next is tangent to the bugs path.)

(b) Find the distance traveled by a bug by the time it meets the other bugs at the center.

SOLN: As time progresses, symmetry suggests that the 4 bugs will occupy the corners of a shrinking and rotating square. Thus we can assume that when the upper right bug is at (r, θ), the upper left but is at $\left(r, \theta + \frac{\pi}{2}\right)$. Compute the slope of the line tangent to the upper right bug's path in two ways: 1) it is $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}r\sin\theta}{\frac{d}{d\theta}r\cos\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$. Also, it is the slope of the line between the two bugs: $\frac{dy}{dx} = \frac{r\sin\theta - r\sin(\theta + \frac{\pi}{2})}{r\cos\theta - r\cos(\theta+\frac{\pi}{2})} = \frac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta}$ Equating these we have

$\frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta}$

Dividing through by $\cos\theta$: $\frac{\frac{dr}{d\theta}\tan\theta + r}{\frac{dr}{d\theta} - r\tan\theta} = \frac{\tan\theta - 1}{1 + \tan\theta} \Leftrightarrow \left(\frac{dr}{d\theta}\tan\theta + r\right)(1+\tan\theta) = \left(\frac{dr}{d\theta} - r\tan\theta\right)(\tan\theta - 1)$

$\Leftrightarrow \frac{dr}{d\theta}(\tan\theta + \tan^2\theta) + r + r\tan\theta = \frac{dr}{d\theta}(\tan\theta + 1) - r\tan^2\theta + r\tan\theta$

$\Leftrightarrow \frac{dr}{d\theta}(\tan^2\theta + 1) = -r(1+\tan^2\theta) \Leftrightarrow \frac{dr}{d\theta} = -r$

This is a separable differential equation $\Leftrightarrow \frac{dr}{r} = -d\theta \Rightarrow \int\frac{dr}{r} = -\int d\theta \Leftrightarrow \ln(r) = -\theta + c \Leftrightarrow r = e^{-\theta+c} = Ae^{-\theta}$

The initial condition is that when $\theta = \frac{\pi}{4}$, $r = \frac{\sqrt{2}}{2}a$, so $A = \frac{\sqrt{2}}{2}ae^{\pi/4}$ and thus $\boxed{r = f(\theta) = \frac{\sqrt{2}}{2}a\exp(\pi/4 - \theta)}$

To find the distance the bug travels we need to know what θ gives $r = 0$, which is $\theta = \infty$ Yikes! No worries, we can do integrals improperly! Length $L = \int\limits_{\pi/4}^{\infty}\sqrt{\left(\frac{d}{d\theta}\frac{\sqrt{2}}{2}a\exp(\pi/4-\theta)\right)^2 + \left(\frac{\sqrt{2}}{2}a\exp(\pi/4-\theta)\right)^2}\,d\theta$

Now $(f(\theta))^2 = (f'(\theta))^2 = \frac{a^2}{2}\exp\left(\frac{\pi}{2} - 2\theta\right)$ so $L = a\int\limits_{\pi/4}^{\infty}\exp\left(\frac{\pi}{4} - \theta\right)d\theta = \lim_{b\to\infty} -a\exp\left(\frac{\pi}{4}-\theta\right)\Big|_{\pi/4}^{b} = a$