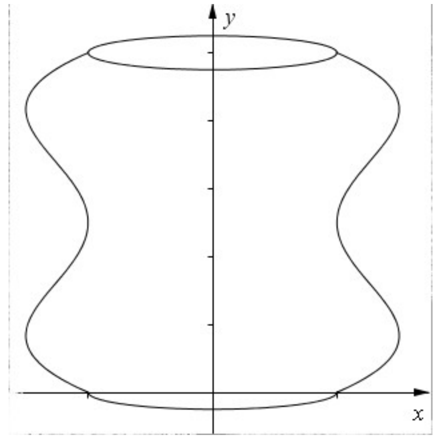
Math 1B **Name (Print):** ______________

Chapter 7 Test Solutions

Write all responses on separate paper. Show your work for credit. Do not use a calculator.

1. (20 points) The curve $x = 2\sin(\pi y) + \cos(2\pi y)$ for $0 \le y \le 1$ generates the volume depicted at right when revolved about the y−axis. If the volume is filled with water, find the minimum work required to pump the water out through the top. Assume all units are MKS (meter, kilogram, second) and that the weight density of water is 9800 Newtons per cubic meter.
   These identities are useful in the integral:
   $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}, \cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$,
   $2\sin(\alpha)\cos(\beta) = \sin(\alpha-\beta) + \sin(\alpha+\beta)$



Let $r = r(y) =$ the radius of the cross-sectional circle at position y - sketch one such circle in the diagram.

$W = \int dW = \int \texttt{dist} \cdot dF = \int (1-y)\,(9800\,dV) = 9800 \int_0^1 (1-y)\pi r^2\,dy = 9800\pi \int_0^1 (1-y)\,(r(y))^2\,dy$

$9800\pi \left[\int_0^1 r^2\,dy - \int_0^1 yr^2\,dy\right]$

$= 9800\pi \left[\int_0^1 4\sin^2(\pi y) + 4\sin(\pi y)\cos(2\pi y) + \cos^2(2\pi y)\,dy - \int_0^1 y\,(2\sin(\pi y) + \cos(2\pi y))^2\,dy\right]$

We'll want to do the second integral by parts, but for that, we'll need the antiderivative of the first integrand.

Start with $\int r^2\,dy = \int (2\sin(\pi y) + \cos(2\pi y))^2\,dy = \int 4\sin^2(\pi y) + 4\sin(\pi y)\cos(2\pi y) + \cos^2(2\pi y)\,dy$

Using the identities given in the hints we get the total weight (good hints!):

$9800\pi \int_0^1 r^2(y)\,dy = 9800\pi \int_0^1 2 - 2\cos(2\pi y) + 2(\sin(-\pi y) + 2\sin(3\pi y) + \dfrac{1+\cos(4\pi y)}{2}\,dy$

$= 9800\pi \left(\dfrac{5y}{2} - \dfrac{1}{\pi}\sin(2\pi y) + \dfrac{2}{\pi}\cos(\pi y) - \dfrac{2}{3\pi}\cos(3\pi y) + \dfrac{1}{8\pi}\sin(4\pi y)\right)\Big|_0^1 = 9800\pi\left(\dfrac{5}{2} - \dfrac{4}{\pi} + \dfrac{4}{3\pi}\right)$

$= 9800\left(\dfrac{5\pi}{2} - \dfrac{8}{3}\right) \approx 2\times 10^4$ $\texttt{N}$, but this will be (times 1 meter) in Joules for our computation.

Now for the second piece we integrate by parts, reusing the antiderivative already found:

$\int_0^1 yf(y)\,dy = y\int f(y)\,dy\Big|_0^1 - \int_0^1 \int f(y)\,dy^2$ $\boxed{\begin{array}{ll} u = y & dv = f(y)dy \\ du = dy & v = \int f(y)dy \end{array}}$

$= \left(\dfrac{5y^2}{2} - \dfrac{y}{\pi}\sin(2\pi y) + \dfrac{2y}{\pi}\cos(\pi y) - \dfrac{2y}{3\pi}\cos(3\pi y) + \dfrac{y}{8\pi}\sin(4\pi y)\right)\Big|_0^1$

$\qquad - \int_0^1 \left(\dfrac{5y}{2} - \dfrac{1}{\pi}\sin(2\pi y) + \dfrac{2}{\pi}\cos(\pi y) - \dfrac{2}{3\pi}\cos(3\pi y) + \dfrac{1}{8\pi}\sin(4\pi y)\right)dy$

$= \dfrac{5}{2} - \dfrac{2}{\pi} + \dfrac{2}{3\pi} - \left(\dfrac{5y^2}{4} + \dfrac{1}{2\pi^2}\cos(2\pi y) + \dfrac{2}{\pi^2}\sin(\pi y) - \dfrac{2}{9\pi^2}\sin(3\pi y) - \dfrac{1}{32\pi^2}\cos(4\pi y)\right)\Big|_0^1 = \dfrac{3}{4} - \dfrac{2}{\pi} + \dfrac{2}{3\pi}$

Putting it all back together, $W = 9800\left[\left(\dfrac{5\pi}{2} - \dfrac{8}{3}\right) - \left(\dfrac{5\pi}{4} - 2 + \dfrac{2}{3}\right)\right] = 9800\left(\dfrac{5\pi}{4} - \dfrac{4}{3}\right)$

Mathematica yields $\int_0^1 9800\pi(1-y)(2\sin(\pi y) + \cos(2\pi y))^2\,dy = \frac{2450}{3}(15\pi - 16)$ We agree!

2. (20 points) Derive a reduction formula for $I_n = \int x^n e^{ax}\,dx$ in terms of I_{n-1}.

SOLN: Using integration by parts with $\boxed{\begin{array}{ll} u = x^n & dv = e^{ax}dx \\ du = nx^{n-1}dx & v = \frac{1}{a}e^{ax} \end{array}}$ we have

$$I_n = \int x^n e^{ax}\,dx = \frac{x^n}{a}e^{ax} - \int nx^{n-1}\frac{1}{a}e^{ax}\,dx = \frac{x^n}{a}e^{ax} - \frac{n}{a}I_{n-1}$$

Use the reduction formula to evaluate $\int_0^1 x^4 e^{ax}\,dx$

SOLN: $\int_0^1 x^4 e^{ax}\,dx = \frac{x^4}{a}e^{ax}\Big|_0^1 - \frac{4}{a}I_3 = \frac{e^a}{a} - \frac{4}{a}\left[\frac{x^3}{a}e^{ax}\Big|_0^1 - \frac{3}{a}I_2\right]$

$$= \frac{e^a}{a} - \frac{4}{a}\left[\frac{e^a}{a} - \frac{3}{a}\left(\frac{e^a}{a} - \frac{2}{a}I_1\right)\right] = \frac{e^a}{a} - \frac{4}{a}\left[\frac{e^a}{a} - \frac{3}{a}\left[\frac{e^a}{a} - \frac{2}{a}\left(\frac{e^a}{a} - \frac{1}{a}\frac{e^a - 1}{a}\right)\right]\right]$$

$$\frac{(a^4 - 4a^3 + 12a^2 - 24a + 24)e^a - 24}{a^5} = \frac{(a(a(a(a-4)+12)-24)+24)e^a - 24}{a^5}$$

3. Let

$$U(x) = \begin{cases} 0 & : x < 0 \\ 1 & : x \geq 0 \end{cases}$$

and then define $\delta(x) = U'(x)$

(a) Show that $\int_{-A}^{A} \delta(x)\,dx = 1$.

SOLN: $\int_{-A}^{A} \delta(x)\,dx = \int_{-A}^{A} \frac{dU}{dx}\,dx = U(A) - U(-A) = 1 - 0 = 1$

(b) Use integration by parts to prove that for any function $v(x)$, $\int_{-A}^{A} v(x)\delta(x)\,dx = v(0)$.

$\boxed{\begin{array}{ll} u = v(x) & dw = \delta(x)\,dx \\ du = v'(x)\,dx & w = U(x) \end{array}} \Rightarrow \int_{-A}^{A} v(x)\delta(x)\,dx = v(x)U(x)\Big|_{-A}^{A} - \int_{-A}^{A} U(x)v'(x)dx = v(0).$

$$= v(A)U(A) - v(-A)U(-A) - \int_0^A v'(x)dx = v(A) - (v(A) - v(0)) = v(0).$$

4. (20 points) Evaluate each integral:

(a) $\int_0^1 \frac{x}{\sqrt{x^2+2x+2}}\,dx = \int_0^1 \frac{x}{\sqrt{(x+1)^2+1}}\,dx$. Let $x+1 = \tan\theta$ so that $x = \tan\theta - 1$, and $dx = \sec^2\theta\,d\theta$.

As $0 \leq x \leq 1$, $\frac{\pi}{4} \leq \theta \leq \arctan(2)$, so the integral becomes,

$\int_{\pi/4}^{\arctan(2)} \frac{\tan\theta - 1}{\sqrt{\tan^2\theta + 1}}\sec^2\theta\,d\theta = \int_{\pi/4}^{\arctan(2)} (\tan\theta - 1)\sec\theta\,d\theta$. Split the integrand for two known integrals

$$\int_{\pi/4}^{\arctan(2)} \sec\theta\tan\theta\,d\theta - \int_{\pi/4}^{\arctan(2)} \sec\theta d\theta = \sec\theta + \ln|\sec\theta + \tan\theta|\Big|_{\pi/4}^{\arctan(2)} = \sqrt{5} - \sqrt{2} + \ln(\sqrt{5}+2) - \ln(\sqrt{2}+1)$$

(b) $\int_2^3 \frac{5}{(x-1)(x^2+2x+2)}\,dx = \int_2^3 \frac{1}{x-1} - \frac{x+3}{x^2+2x+2}\,dx = \ln(2) - \frac{1}{2}\int_2^3 \frac{2x+2}{x^2+2x+2} + \frac{4}{(x+1)^2+1}\,dx$

$\ln 2 - \frac{1}{2}(\ln(17) - \ln(10)) - 2\arctan(x+1)\Big|_2^3 = \frac{1}{2}\left(\ln 4 - \ln\frac{17}{10}\right) - 2(\arctan(4) - \arctan(3))$

$$= \frac{1}{2}\left(\log\left(\frac{40}{17}\right) + 4\tan^{-1}(3) - 4\tan^{-1}(4)\right)$$

5. (20 points) Determine whether each integral is convergent or divergent. If it's convergent, evaluate it.

(a) $$\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx = \lim_{b\to\infty}\int_1^b u^{-1/4}du = \lim_{b\to\infty}\frac{4}{3}u^{3/4}\Big|_1^b = \lim_{b\to\infty}\frac{4}{3}\left(b^{3/4}-1\right) = \infty \Rightarrow \text{divergent.}$$

(b) $$\int_0^1 \frac{5}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-}\int_0^b \frac{5}{\sqrt{1-x^2}}\,dx$$

Substituting $x = \cos\theta, dx = -\sin\theta d\theta$, we have

$\lim_{b\to 1^-}\int_0^b \frac{5}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/2} 5\,d\theta = \frac{5\pi}{2}$

Note the double negative in reversing the bound with the new variable.