Math 1B –Test 5 – Spring '10 Name______________________________

Directions: Show work for credit. Write all responses on separate paper.

1. Find the radius of convergence and interval of convergence for each series:
   a. $\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n+1)!}$ b. $\sum_{n=1}^{\infty} \frac{n(x-3)^{2n}}{n^2+1}$ c. $\sum_{n=1}^{\infty} \frac{(2n)!x^{2n+1}}{2\cdot 6\cdot 10\cdots(4n-2)}$

2. Find a power series representation for the function and determine the interval of convergence. Write out the first 3 non-zero terms in each series.
   a. $f(x)=\frac{5}{1+x^3}$ b. $f(x)=\ln(x^2+9)$ c. $f(x)=\arctan\left(\frac{x}{2}\right)$

3. Consider $f(x)=\frac{\arctan(2x)}{x}$
   a. How can $f(0)$ be defined so that the function is continuous where $x = 0$?
   b. Using this definition for $f(0)$, develop, step by step, the Maclaurin series for
   $$f(x)=\sum_{n=0}^{\infty}(-1)^n 2^{2n+1}\frac{x^{2n}}{2n+1}$$
   c. Find a Maclaurin series for $\int_0^x f(t)\,dt$ and use this to approximate $\int_0^{1/10}\frac{\arctan(2x)}{x}dx$ to the nearest thousandth.

4. Let $f_n(x)=\frac{\sin(x/n)}{n}$
   a. Show that the series $\sum f_n(x)$ diverges for $x = n\pi/2$, n an integer.
   b. Show that $\sum f_n'(x)$ converges for all x.
   c. For what values of x does the series $\sum f_n''(x)$ converge?

5. Find the Taylor series for f centered at $x = 4$ if $f^{(n)}(4)=\frac{(-1)^n n!}{2^n(2n+1)}$.

   What is the radius of convergence of this Taylor series?

6. Find the Maclaurin series for $\cosh(x)=\frac{e^x+e^{-x}}{2}$. What is the radius of convergence?

7. Find the Taylor series for $f(x)=\frac{1}{x}$ centered around $a = -2$.

   This means you'll need to find a formula for the Taylor coefficient.

8. Find a binomial series for $f(x)=(1+x)^{1/3}$ and use this to approximate $\sqrt[3]{1729}=\sqrt[3]{1+12^3}$.

   You need only work out the first two non-zero terms.

Math 1B –Test 5 Solutions – Spring '10

1. Find the radius of convergence and interval of convergence for each series:

   a. $\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n+1)!}$ SOLN: By the ratio test, $\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+3)!}\frac{(2n+1)!}{x^{2n}}\right|=\lim_{n\to\infty}\left|\frac{x^2}{(2n+3)(2n+2)}\right|=0$ for any fixed x, so the radius is infinite and the interval is all real numbers.

   b. $\sum_{n=1}^{\infty}\frac{n(x-3)^{2n}}{n^2+1}$ SOLN: By the ratio test,
   $\lim_{n\to\infty}\left|\frac{(n+1)(x-3)^{2n+2}}{(n+1)^2+1}\frac{n^2+1}{n(x-3)^{2n}}\right|=\lim_{n\to\infty}(x-3)^2\left|\frac{n+1}{n}\right|\left|\frac{n^2}{n^2+2n+2}\right|=(x-3)^2<1\Leftrightarrow|x-3|<1$ so the radius of convergence is 1 and the endpts of the interval are 2 and 4. These need to be checked.
   So at x = 2 or x = 4 we have $(x-3)^{2n}=1$ which means the series are $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$, which is divergent by comparison with the harmonic series. So the interval of convergence is (2,4).

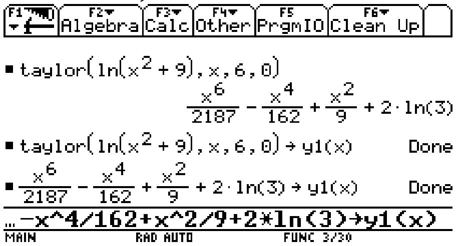
   c. $\sum_{n=1}^{\infty}\frac{(2n)!x^{2n+1}}{2\cdot6\cdot10\cdots(4n-2)}$ SOLN: By the ratio test,
   $\lim_{n\to\infty}\left|\frac{(2n+2)!x^{2n+3}}{2\cdot6\cdot10\cdots(4n-2)(4n+2)}\frac{2\cdot6\cdot10\cdots(4n-2)}{(2n)!x^{2n+1}}\right|=\lim_{n\to\infty}x^2\left|\frac{(2n+2)(2n+1)}{4n+2}\right|=\infty$ unless x = 0. So the radius of convergence is 0 and there is no interval of convergence, only the point x = 0.

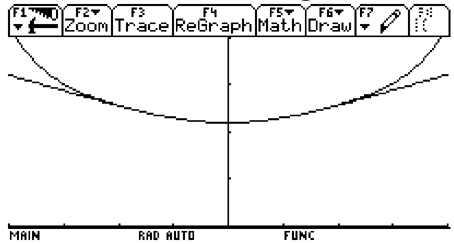
2. Find a power series representation for the function and determine the interval of convergence.
   Write out the first 3 non-zero terms in each series.

   a. $f(x)=\frac{5}{1+x^3}=5\frac{1}{1-(-x^3)}=5\sum_{n=0}^{\infty}(-x^3)^n=5\sum_{n=0}^{\infty}(-1)^n x^{3n}=5-5x^3+5x^6+\cdots$
   The interval of convergence is (–1,1).

   b. $f(x)=\ln(x^2+9)=\int\frac{2x}{x^2+9}dx=\int\frac{2x}{9}\frac{1}{1-\left(-\frac{x^2}{9}\right)}dx=\int\frac{2x}{9}\sum_{n=0}^{\infty}\left(-\frac{x^2}{9}\right)^n dx$
   $=2\sum_{n=0}^{\infty}\frac{(-1)^n}{9^{n+1}}\int x^{2n+1}dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{9^{n+1}(n+1)}=c+\frac{1}{9}x^2-\frac{1}{162}x^4+\frac{1}{2187}x^6+\cdots$
   For the graphs to agree at zero we need to choose c = ln(9).
   The ratio test shows that $\lim_{n\to\infty}\left|\frac{x^{2n+4}}{9^{n+2}(n+2)}\frac{9^{n+1}(n+1)}{x^{2n+2}}\right|=\lim_{n\to\infty}\left|\frac{x^2}{9}\right|<1\Leftrightarrow|x|<3$. At ±3, the series is the alternating harmonic series, which is convergent. So the interval of convergence is [–3,3].

To be sure, here are some screen shots of the TI-92 illustrating these functions and their graphs.

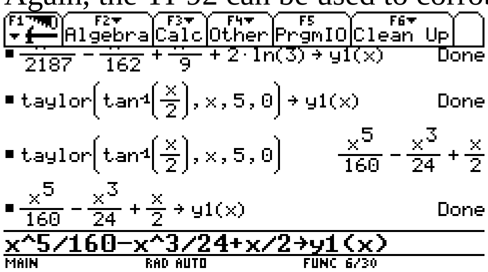


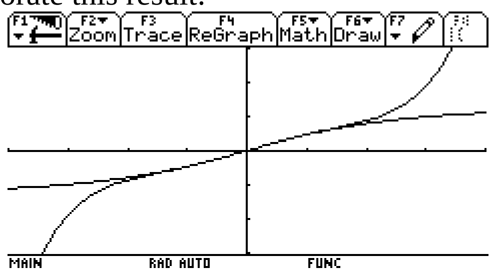


c. $$f(x) = \arctan\left(\frac{x}{2}\right) = \int \frac{d}{dx}\arctan\left(\frac{x}{2}\right)dx = \frac{1}{2}\int \frac{1}{1-\left(-\frac{x^2}{4}\right)}dx = \frac{1}{2}\int\sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^n dx$$

$$= \frac{1}{2}\int\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{4^n}dx = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}}\int x^{2n}dx = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{2n+1}(2n+1)} = \frac{1}{2}x - \frac{1}{24}x^3 + \frac{1}{160}x^5 + \cdots$$

The ratio test shows that $\lim_{n\to\infty}\left|\frac{x^{2n+3}}{2^{2n+3}(2n+3)}\frac{2^{2n+1}(2n+1)}{x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{4}\right| < 1 \Leftrightarrow |x| < 2$. At ±2, the series is an alternating series that satisfies the alternating series test, so the interval of convergence is [–2,2]. Again, the TI-92 can be used to corroborate this result:





3. Consider $f(x) = \frac{\arctan(2x)}{x}$

a. How can $f(0)$ be defined so that the function is continuous where $x = 0$?

SOLN: $\lim_{x\to 0}\frac{\arctan(2x)}{x} = \lim_{x\to 0}\frac{\frac{2}{1+4x^2}}{1} = 2$, so define $f(0) = 2$ to remove the discontinuity.

b. Using this definition for $f(0)$, develop the Maclaurin series for $f(x) = \sum_{n=0}^{\infty}(-1)^n 2^{2n+1}\frac{x^{2n}}{2n+1}$

SOLN: $$f(x) = \frac{1}{x}\int\frac{d}{dx}\arctan(2x)\,dx = \frac{2}{x}\int\frac{1}{1-(-4x^2)}dx = \frac{2}{x}\int\sum_{n=0}^{\infty}(-4x^2)^n dx$$

$$= \frac{2}{x}\sum_{n=0}^{\infty}(-4)^n\int x^{2n}dx = \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+1}x^{2n}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{2(2x)^{2n}}{2n+1} = 2 - \frac{8}{3}x^2 + \frac{32}{5}x^4 + \cdots$$

c. Find a Maclaurin series for $\int_0^x f(t)\,dt$ and use this to approximate $\int_0^{1/10}\frac{\arctan(2x)}{x}dx$ to the nearest thousandth.

$$\int_0^x\sum_{n=0}^{\infty}(-1)^n\frac{2(2t)^{2n}}{2n+1}dt = \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+1}}{2n+1}\int_0^x t^{2n}dt = \sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n+1}}{(2n+1)^2} = 2x - \frac{8}{9}x^3 + \frac{32}{25}x^5 - \frac{512}{81}x^7 + \cdots$$

The radius of convergence here is ½, so $\int_0^{1/10} \frac{\arctan(2x)}{x}dx$ can be evaluated as

$$\sum_{n=0}^{\infty}(-1)^n \frac{(0.2)^{2n+1}}{(2n+1)^2} = 2(0.2) - \frac{8}{9}(.008) + \frac{32}{25}0.00032 - \cdots \approx 0.4 - 0.007\overline{1} + 0.00041 \approx 0.393$$

4. Let $f_n(x) = \frac{\sin(x/n)}{n}$

   a. Show that the series $\sum f_n(x)$ diverges for $x = n\pi/2$, n an integer.

   SOLN: $f_n\left(\frac{n\pi}{2}\right) = \frac{\sin(\pi/2)}{n} = \frac{1}{n}$ so $\sum f_n(x) = \sum 1/n$ is the (divergent) harmonic series.

   b. Show that $\sum f_n^{/}(x)$ converges for all x.

   SOLN: $\left|f_n'(x)\right| = \left|\frac{\cos(x/n)}{n^2}\right| \le \frac{1}{n^2}$ and so the series converges absolutely (p-series with $p = 2$).

   c. For what values of x does the series $\sum f_n''(x)$ converge?

   $\left|f_n''(x)\right| = \left|\frac{\sin(x/n)}{n^3}\right| \le \frac{1}{n^3}$ converges for all x.

5. Find the Taylor series for f centered at $x = 4$ if $f^{(n)}(4) = \frac{(-1)^n n!}{2^n(2n+1)}$.

   What is the radius of convergence of this Taylor series?

   SOLN: $c_n = \frac{f^{(n)}(4)}{n!} = \frac{(-1)^n}{2^n(2n+1)} \Rightarrow f(x) = \sum_{n=0}^{\infty} c_n(x-4)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n(2n+1)}(x-4)^n$

   By the ratio test, $\lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{2^{n+1}(2n+3)}\frac{2^n(2n+1)}{(x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{x-4}{2}\right| < 1 \Leftrightarrow |x-4| < 2$ so the radius of convergence is 2.

6. Find the Maclaurin series for $\cosh(x) = \frac{e^x + e^{-x}}{2}$. What is the radius of convergence?

   SOLN $\cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ Note that all the odd order terms cancel out.

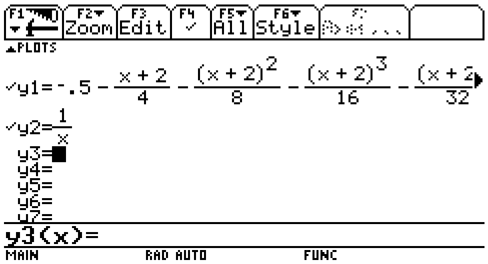
   The series converges for all x.

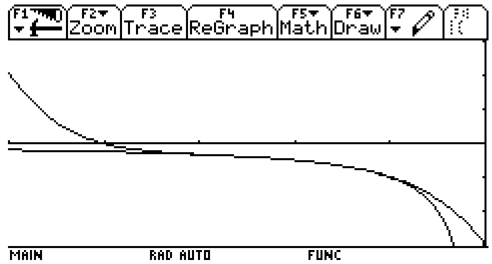
7. Find the Taylor series for $f(x) = \frac{1}{x}$ centered around $a = -2$.

   This means you'll need to find a formula for the Taylor coefficient.

   SOLN: $f'(x) = -\frac{1}{x^2}, f''(x) = \frac{2}{x^3}, f'''(x) = -\frac{3\cdot 2}{x^4}, \ldots \Rightarrow f^{(n)}(x) = (-1)^n\frac{n!}{x^{n+1}}$

   so that $c_n = \frac{f^{(n)}(-2)}{n!} = \frac{(-1)^n}{(-2)^{n+1}} = -\frac{1}{2^{n+1}}$ and the Taylor series is $\frac{1}{x} = -\sum_{n=0}^{\infty}\frac{(x+2)^n}{2^{n+1}}$





8. Find a binomial series for $f(x)=(1+x)^{1/3}$ and use this to approximate $\sqrt[3]{1729}=\sqrt[3]{1+12^3}$.
   You need only work out the first two non-zero terms.

   SOLN: $f(x)=(1+x)^{1/3}=\sum_{n=0}^{\infty}\binom{1/3}{n}x^n=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots$

   So $\sqrt[3]{1729}=\sqrt[3]{1+12^3}=12\sqrt[3]{1+\frac{1}{1728}}\approx 12\left(1+\frac{1}{3\cdot 1728}\right)=12+\frac{1}{432}=12.0023\overline{148}$

   Note: This compares closely with $\sqrt[3]{1729}\approx 12.0023143684$