How Things Float

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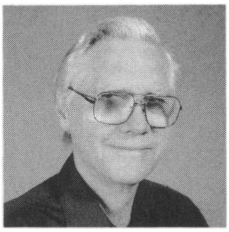

ABSTRACT S. M. Ulam once asked if spheres are the only homogeneous bodies that can float in every orientation. Here an affirmative answer is given for a special class of bodies of revolution that seemed likely to contain counterexamples. Traditionally, the floating bodies of most interest have been ships, which are not homogeneous. However, homogeneous bodies are found to float in surprising ways. Bodies with symmetry often float asymmetrically. Thus, circular cylinders can float with the axis of symmetry tilted away from vertical at strange angles. A cube or regular tetrahedron can float with neither a vertex, an edge center, nor a face center down. Indeed, like the circular cylinder, a cube may have infinitely many indifferently stable floating orientations that are not isolated, but form a one-parameter family.

1. Introduction. The stable orientations of a floating body present a classic problem of hydrostatics. Naval architects use a well-known stability condition (Theorem 1 below) to design ship hulls that float stably under a variety of loading conditions. With the exception of icebergs (Section 4), floating homogeneous bodies have little practical importance. However they have interesting mathematical properties. A simple law connects the stable floating positions of congruent homogeneous bodies of densities ρ and $1 - \rho$ (Theorem 2). Spheres may be the only homogeneous bodies that can float in all orientations (Ulam [8]), but that has never been proved or disproved (Section 3). Theorem 4 proves it for a special class of bodies. A homogeneous body with an axis of rotational symmetry always has an equilibrium orientation with the axis vertical, but the equilibrium may be unstable. For example, only the shortest semiaxes of a homogeneous ellipsoid provide stable equilibria (Section 6). Depending on its dimensions and density, a circular cylinder may float with its axis vertical, horizontal, or tilted at some special angle away from vertical (Section 8). Regular polyhedra have many symmetry axes (through vertices, face centers, and edge centers) but, again, these orientations need not be stable. All of them are unstable for a tetrahedron of density $1/2$ and so this tetrahedron floats in a way that is not easily guessed (Section 5). Likewise, for some values of ρ, the cube floats in an asymmetric way. In fact, like many circular cylinders, some cubes do not float in isolated stable orientations; instead they have infinitely many indifferently stable orientations, forming a one-parameter family (Section 10).

2. Potential and metacenter. It is convenient to use coordinates fixed in the body. The problem then is to predict where the waterline will be on the body when it is floating stably. Let n be a unit normal to the water surface, directed up from the water. It suffices to find n since then Archimedes' principle determines the waterline.

Stable positions for a floating body occur at minima of a potential energy function $U(n)$. $U(n)$ is the work done to bring the body into a position with n

upward and with the waterline at the level where the body displaces its own weight in water. The work done against gravity is mgS_1 where m is the mass of the body, g the acceleration of gravity, and S_1 the height of the center of gravity G of the body. The work done against buoyancy forces is just the work required to raise the particles of the displaced water up to the water level. This work is mgS_2 where S_2 is the depth of the centroid B of the submerged part of the body. B is called the *center of buoyancy*. Thus

$$U(n) = mg(S_1 + S_2) = mgn \cdot (G - B). \tag{1}$$

Since the gravity force acts downward from the center of gravity and the buoyancy force acts upward from the center of buoyancy, the line joining B and G must have the vertical direction n in equilibrium. Some equilibrium directions are maxima or saddle points of the potential function $U(n)$ and represent unstable equilibrium orientations of the body.

If n_0 is an equilibrium direction, the behavior of $S(n) \equiv n \cdot (G - B)$ near n_0 determines the type of equilibrium. To perturb n away from n_0, rotate the body through a small angle α about an axis with unit direction vector t normal to n_0. Then $n = n_0 \cos\alpha + (n_0 \times t)\sin\alpha$. If $S(n)$ has continuous partial derivatives of order 3 at n_0,

$$S(n) = S(n_0) + \text{const}\,\alpha^2 + O(\alpha^3).$$

Buoyancy forces will tend to move the body back to orientation n_0 if the constant in the quadratic term is positive, or further away from n_0 if the constant is negative. The main theorem about floating bodies expresses this constant in terms of a point M, called the metacenter, which is defined as follows.

For a given equilibrium direction n_0, let V denote the volume of the submerged part of the body and let A denote the two-dimensional figure in which the plane of the waterline intersects the body. Let I denote the moment of inertia of A about an axis with direction t $(t \cdot n_0 = 0)$ and passing through the centroid of A. Then the *metacenter* for n_0 and t is the point

$$M = B + n_0 I/V \tag{2}$$

lying on the (vertical) line through G and B at height I/V above B.

THEOREM 1. *When n is rotated away from equilibrium direction n_0, through a small angle α about axis t,*

$$n \cdot (G - B) = S(n) = S(n_0) + \frac{n_0 \cdot (M - G)\alpha^2}{2} + O(\alpha^3). \tag{3}$$

Buoyancy forces act to move the body back to orientation n_0 if M lies above G and away from n_0 if M lies below G.

The idea of metacenter and its connection with stability date back to Huyghens. Theorem 1 appears in many books on mechanics ([**1**], [**9**]) and naval architecture ([**6**]).

Since the metacenter depends on t, an equilibrium orientation n_0 may be stable with respect to some perturbation axes t and unstable for others (saddle-point equilibrium). If the principal moments of inertia of A about its centroid are I_1 and I_2, with $I_1 \leqslant I_2$, let the metacenters for rotations about the corresponding principal axes be M_1 and M_2. Then n_0 is a minimum (stable point), saddle point, or maximum of $S(n)$ if G lies below M_1, between M_1 and M_2, or above M_2. In naval architecture M_1 and M_2 are the metacenters for rolling and pitching.

B, M_1, and M_2 are determined entirely by the shape of the submerged part of the body but G may be moved about by redistributing cargo. To make a ship float stably in a particular orientation n_0, its mass must be distributed to put G below M_1. For instance a small boat, originally stable, can become unstable if its passenger stands up. A vessel partly filled with liquid (a tanker, leaky ship, or half-empty bottle) requires a more complicated treatment; its cargo includes liquid that moves as the body is perturbed.

In what follows the body is a solid made from homogeneous material of density $\rho < 1$. This is not an important case for naval architecture but it contains some mathematical curiosities. Symmetry axes of bodies like regular polyhedra, cylinders, or cones are obvious equilibrium directions. The kinds of stability that these bodies exhibit are often surprising.

3. Indifferent stability. Because of its symmetry, a homogeneous sphere has the same potential $U(n) = mgn \cdot (G - B)$ for all orientations n. Every n is then an equilibrium direction. But these equilibria are indifferently stable. That is, the perturbed body tends neither to return to its original orientation nor to move away from it. Likewise a long circular cylinder has the same potential $U(n)$ for all directions n normal to the cylinder's axis. These directions are equilibria, indifferently stable with respect to rolling although stable with respect to pitching.

Ulam [**8**] asked, "A solid S of uniform density ρ has the property that it will float in equilibrium (without turning) in water in every orientation. Must S be a sphere?" No nonspherical body with this property is known. H. Auerbach [**2**] settled a similar problem for cylindrical bodies by constructing noncircular cylinders that float indifferently stably with respect to rolling, in the manner of a circular cylinder. FIGURE 1 is the cross section of one of Auerbach's indifferently stable cylinders. Chords drawn in FIGURE 1 represent waterlines for different floating positions. These chords cut the figure in two halves of equal area ($\rho = 1/2$). Different heart-shaped cross sections work for other densities and there are other solutions that are not heart-shaped.

For a body to have constant $U(n)$, as Ulam required, (1) shows $B = G - hn$ with h a constant. As n varies, the center of buoyancy remains on a sphere of

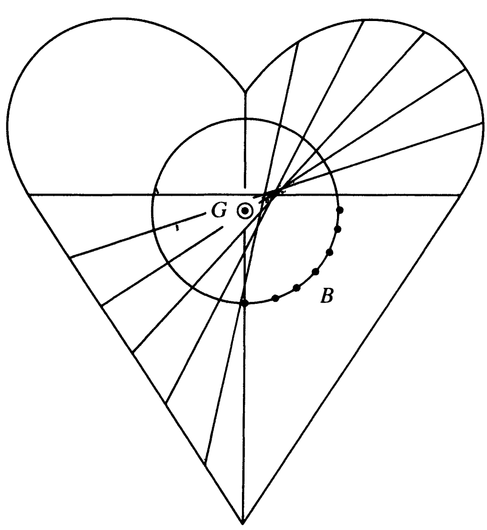

FIG. 1. Cross section of indifferently stable floating cylinder of density 1/2. Chords are possible waterlines. Centers of buoyancy B lie on a circle about G.

radius h and center G. Moreover, (3) requires $M = G$ and then (2) shows $I = hV$, a constant for all n and t.

In Auerbach's cylinder problem, the same reasoning requires that centers of buoyancy lie on a circle, as FIGURE 1 shows. To achieve constant moment of inertia I for rolling, the waterline chords in FIGURE 1 all have the same length. These constraints supply the differential equation that determined the shape of FIGURE 1.

When ρ is small, the body floats with small submerged volume and, in the limit as $\rho \to 0$, as though resting on a flat table. Ulam also asked, "If a body rests in equilibrium in every position on a flat horizontal surface is it a sphere?" The answer to that question must be "yes," at least for bodies that touch the flat surface at only one point $P(n)$. For, as $\rho \to 0$, B must approach $P(n)$; then boundary points $P(n)$ all lie on a sphere. For a purely geometric argument, not using the potential or metacenter, see Montejano [**4**].

4. Capsized bodies. The stable equilibrium positions of a floating body made from homogeneous material of density ρ will ordinarily depend on ρ. If equilibria are known for $\rho \leqslant 1/2$, the following theorem will immediately supply equilibria for $\rho \geqslant 1/2$.

THEOREM 2. *Consider a homogeneous floating body that may have either density ρ or $\rho' = 1 - \rho$. Let the potential functions of the body with densities ρ and ρ' be $U(n)$ and $U'(n)$. Then $U'(-n) = U(n)$.*

Proof. Let the body have volume K. The body with density ρ or ρ' displaces water volumes $V = \rho K$ or $V' = \rho' K$. For a given direction vector n let B denote the center of buoyancy of the body with density ρ. Write B' for the centroid of the part of the body above the water level. This part has volume $K - V = V'$ and so B' is the center of buoyancy when the body with density ρ' is oriented with direction vector $-n$.

From $KG = VB + V'B'$ one finds $V(G - B) + V'(G - B') = 0$ and then

$$-n \cdot (G - B')V' = n \cdot (G - B)V. \tag{4}$$

Now (1) and (4) show that $U'(-n) = U(n)$. ■

An immediate consequence of Theorem 2 is

COROLLARY. *If n_0 is an equilibrium direction for a body of density ρ then $-n_0$ is one for the same body with density $1 - \rho$. The two equilibria are alike (both minima, both maxima, or both saddle points). Bodies with $\rho = 1/2$ float as stably capsized as right-side-up.*

FIGURE 2 illustrates the corollary by showing stable positions for a body with densities ρ and $1 - \rho$.

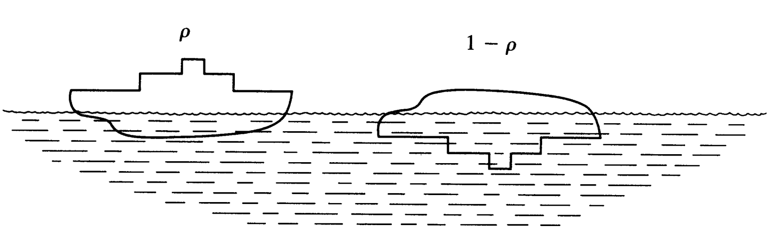

FIG. 2. Stable floating positions for congruent homogeneous bodies of density ρ and $\rho = 1 - \rho$.

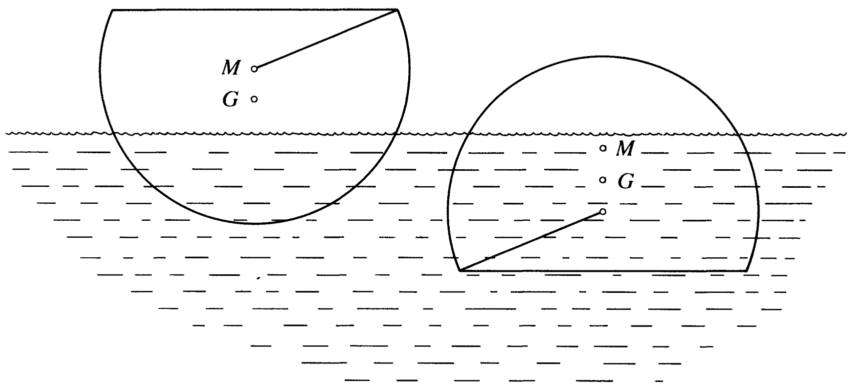

FIG. 3. Two stable orientations of truncated spheres.

For another example, consider a sphere truncated by a plane cut. In one equilibrium position the plane cut is horizontal and above the water level (Figure 3a). The submerged part of the body is spherical and so, as Section 2 remarked, the metacenter M lies at the center of the sphere. Truncating the sphere has moved G from the center M of the sphere to a lower point. Then this equilibrium is stable for any ρ. But now the corollary shows that truncated spheres of all densities also float stably when capsized, with the cut submerged (Figure 3b).

The corollary can help show how icebergs float. Ice has density about $\rho = .9$ relative to sea water. Consider, instead, a fictitious iceberg with density $\rho' = 1 - \rho = .1$. This light body may be expected to float in nearly the orientation it would assume resting on a flat plane. Then the real iceberg floats in the capsized orientation. For example, an iceberg with a large flat face is likely to float face-up. Of course, as Figure 3b illustrates, it may also float in other orientations, although not as stably.

5. Counting equilibria. In looking for equilibrium points a useful identity relating the numbers of maxima, minima, and saddle points is

$$N(\max) + N(\min) = N(\text{saddle}) + 2, \tag{5}$$

where $N(\cdots)$ is read "number of...". Equation (5) is a topological identity that applies to functions $U(n)$ with continuous second derivatives, defined on the sphere (see [5] and [7]). It assumes equilibrium points are isolated and only of the three kinds indicated. Thus, near an equilibrium point, $U(n)$ must agree to terms of second order with a non-degenerate quadratic form in local coordinates.

The identity provides a check that all equilibrium points have been found and properly classified. For example, the directions of the six semiaxes of a homogeneous ellipsoid are obvious equilibrium points. In Section 6, these are shown to be the only equilibria; there are two maxima, two minima, and two saddle points, in agreement with (5).

The identity (5) has an interesting application to the equilibria of a floating regular polyhedron. The directions from the polyhedron's center to the vertices, face centers, and edge centers are obvious equilibrium directions because of symmetry. Indeed, in the limit as $\rho \to 0$, the body can touch the water surface only on a face, edge, or vertex, and these equilibria are the only possibilities. Moreover their numbers satisfy Euler's identity

$$N(\text{faces}) + N(\text{vertices}) = N(\text{edges}) + 2. \tag{6}$$

With the light regular polyhedron resting on the water surface as though on a table, each face, vertex, or edge provides a minimum, maximum, or saddle point of $U(n)$ (this need not be true of irregular polyhedra) and (5) is an instance of (6).

One might expect directions along axes of symmetry to be the only equilibria even when $\rho > 0$. Axes through faces and vertices are axes of 3-fold symmetry or more; then $M_1 = M_2$ for these axes and their equilibria can be only maxima or minima. Axes through edge centers have only 2-fold symmetry and so might give the saddle-point equilibria that (5) demands. However, the situation can be more complicated. A regular tetrahedron with $\rho = 1/2$ provides a quick example. The corollary to Theorem 2 shows that it floats as stably face-down as vertex-down; indeed the metacenters, obtained from (2), show both orientations to be maxima of $U(n)$. Moreover the orientations with edge down also turn out to be maxima. Then other orientations, not along symmetry axes, will minimize $U(n)$ and even more will provide saddle points.

FIGURE 4a will help find the missing equilibria. It is convenient to represent each n by the point where the ray from G, with direction n, crosses a face of the tetrahedron (gnomonic projection). The corollary to Theorem 2 states that equilibrium directions come in pairs, represented by diametrically opposite points, such as vertex P_4 and face center F, or the edge centers E and E'. The plane of P_4, E', P_3, F, E, and G is a plane of mirror symmetry of the tetrahedron. Then maxima and minima of $U(n)$, with n restricted to vectors lying in the plane, occur only at equilibrium points. Since P_3, F, and E are maxima there must also be two minima, Q between P_3 and F, and R between F and E. To locate Q exactly, note that the plane of P_1, E', P_2, E is another plane of mirror symmetry, and hence a waterline

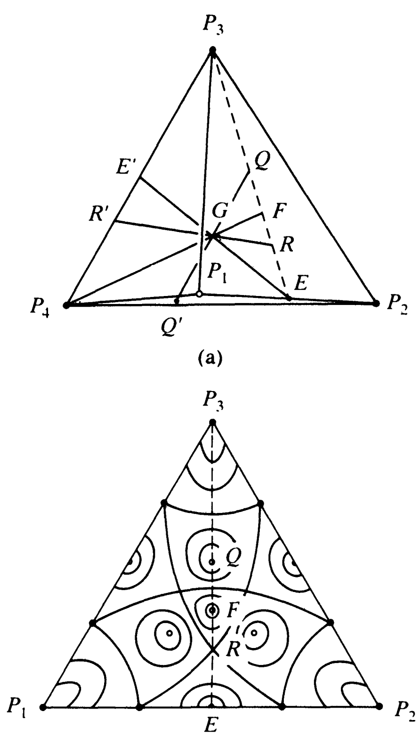

FIG. 4. Equilibrium points and equipotential lines on a regular tetrahedron of density 1/2.

of an equilibrium orientation of the tetrahedron. The normal to this plane is parallel to edge P_4P_3 and represented by Q, the midpoint of the dotted altitude P_3E of face $P_1P_2P_3$. A metacenter calculation shows Q to be a stable point. The point R is more complicated but, lying between F and E, R is diametrically opposite a point R' on edge P_3P_4. Thus the missing equilibria include 12 minima like Q, on midpoints of face-altitudes, 12 points like R, and 12 like R'. Equation (5) will be satisfied if R and R' are saddle points and there are no further equilibria. The laborious calculations, required to locate R and R', find their type of equilibrium, and verify that all equilibria have been found, have not been made. Figure 4b is only an artist's conception of the lines of constant $U(n)$ on a face.

The stable orientations of a tetrahedron of density $1/2$ are related to symmetry because they hold one edge vertical. However, with ρ slightly different from $1/2$, these orientations are no longer equilibria and the body must float asymmetrically.

6. Ellipsoid. The stability of a floating ellipsoid will be determined by first showing how compressing a body affects its stability. Consider any floating body F made of homogeneous material of density ρ. If n_0 is an equilibrium position for F, introduce cartesian body-coordinates x, y, z with the z-axis in the direction n_0. A transformation

$$x' = ax, \qquad y' = by, \qquad z' = cz \tag{7}$$

compresses F into a body F'. Volumes V and V' of corresponding regions in F and F' are related by

$$V' = abcV.$$

If F' is also made of homogeneous material of density ρ, the compression (7) maps the waterline of F to the waterline of F' for the same direction n_0. Centers of gravity G and buoyancy B of F also map to the corresponding centers G' and B' of F'. Then n_0 is also an equilibrium direction of F'. However F and F' may have different types of equilibrium at n_0 because (7) does not map metacenters into metacenters.

Let M and M' be the metacenters of F and F' for small rotations of n about the x-axis. These metacenters are related because the waterline figures A, A' of F, F' have moments of inertia I, I' satisfying $I' = ab^3I$. Then (2) shows

$$n_0 \cdot (M' - B') = I'/V' = ab^3I/(abcV) = \frac{b^2}{c} n_0 \cdot (M - B).$$

The stability condition $0 < n_0 \cdot (M' - G')$ for F' is equivalent to $n_0 \cdot (G' - B') < n_0 \cdot (M' - B')$, or

$$\frac{b^2 n_0 \cdot (M - B)}{c^2 n_0 \cdot (G - B)} > 1. \tag{8}$$

Now, in particular, let F be a unit sphere so that the compression (7) produces an ellipsoid F', with semiaxes a, b, c, floating with c-semiaxis vertical. Because the sphere is indifferently stable, (3) shows that all its metacenters M coincide with G. The condition (8) for stability with respect to small rotations about the a-semiaxis, starting from a configuration with n_0 along a c-semiaxis, reduces to $c < b$. Similarly F' will be stable with respect to small rotations about a b-semiaxis if $c < a$. It follows that the shortest semiaxes of F' are stable directions n, the longest semiaxes maximize the potential $U(n)$, and the middle-sized semiaxes provide saddle points.

If the semiaxes a, b, c of an ellipsoid F are all different, they provide six equilibrium directions, two each of minima, maxima, and saddle points. Equation (5) suggests that these may be the only equilibrium directions. A proof that there are no other equilibrium directions follows.

Consider the ellipsoid $(x'/a)^2 + (y'/b)^2 + (z'/c)^2 = 1$ obtained from a unit sphere $x^2 + y^2 + z^2 = 1$ by the compression (7). Suppose $a < b < c$. Imagine the ellipsoid floating with upward direction $n' = (\lambda', \mu', \nu')$; let $\lambda'x' + \mu'y' + \nu'z' = d'$ denote the water surface. On the unit sphere the corresponding water surface is $\lambda x + \mu y + \nu z = d$ where $n = (\lambda, \mu, \nu)$ is a unit vector with the direction of $(\lambda'a, \mu'b, \nu'c)$. The center of buoyancy B of the sphere lies below the origin G, in the direction of $-n$, say $B = -(\beta\lambda'a, \beta\mu'b, \mu\nu'c)$. Since affine transformations carry centroids into centroids, the center of buoyancy of the ellipsoid is at

$$B' = -\left(\beta a^2\lambda', \beta b^2\mu', \beta c^2\nu'\right).$$

In order for n' to be an equilibrium direction of the ellipsoid, n' and B' must have the same direction. Since a^2, b^2, c^2 are all different, the only possibilities for n' are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$, the semiaxis directions.

7. Body of revolution. For bodies of revolution, $U(n)$ simplifies to a function $u(\theta)$ of just the angle θ between n and the axis a of revolution $(0 \leqslant \theta \leqslant \pi)$. The axial directions $n = \pm a$ are obvious equilibria. Other equilibrium directions n are not isolated, but lie on circles of constant θ. Then (5) does not apply.

For equilibria other than $n = \pm a$ it is convenient to use the term *pitch* for a rotation about a horizontal axis p that is normal to a. Then a *roll* is a rotation about a horizontal axis normal to p, i.e., an axis in the plane of n and a. Because the plane of n and a is a plane of mirror symmetry of the waterline figure A, the pitch and roll axes are principal directions for A. Then M_1 and M_2 are the metacenters for pitching and rolling (although perhaps not in that order).

THEOREM 3. *Consider a homogeneous floating body of revolution with axial direction vector a. At an equilibrium direction $n_0 \neq \pm a$, the roll metacenter M_r is at the center of gravity G.*

Proof. Introduce body coordinates x, y, z with z axis along the axis of rotation and x-axis in the plane of a and n_0. Write $n_0 \cdot a = \cos\theta_0$. Then $n_0 = (\sin\theta_0, 0, \cos\theta_0)$. A roll is a rotation about the x-axis. A roll of size α moves the vertical direction from n_0 to $n = (\sin\theta_0, \cos\theta_0 \sin\alpha, \cos\theta_0\cos\alpha)$. Then $n \cdot a = \cos\theta = \cos\theta_0 \cos\alpha$ and, for small α, $\theta = \theta_0 + (\alpha^2/2)\cot\theta_0 + O(\alpha^4)$. The potential at n is

$$U(n) = u(\theta) = u(\theta_0) + \frac{du}{d\theta}(\theta - \theta_0) + O\left(|\theta - \theta_0|^2\right).$$

At n_0, $du/d\theta = 0$ because n_0 is an equilibrium direction. Then $U(n) = U(n_0) + O(|\theta - \theta_0|^2) = U(n_0) + O(\alpha^4)$ and (3) shows $M_r = G$. ■

Theorem 3 shows that a body of revolution is indifferently stable with respect to small roll perturbations. This result can also be deduced from the observation that the body is indifferently stable to rotations about its axis a of symmetry. For the sake of simplicity, an equilibrium of a body of revolution will be called stable if it is pitch-stable.

COROLLARY. *An equilibrium $n_0 \neq \pm a$ of a body of revolution is stable iff I_p and I_r, the moments of inertia of A for pitching and rolling, satisfy $I_r < I_p$.*

The corollary follows immediately from (2) and Theorems 1 and 3.

An ellipsoid with two semiaxes equal is an ellipsoid of revolution, a spheroid. The results of Section 6 show that the rotation axis a must be either vertical ($\theta = 0$ or π) or horizontal ($\theta = \pi/2$) in any equilibrium position of a spheroid. Oblate spheroids are stable with a vertical and prolate spheroids are stable with a horizontal. Other bodies of revolution can have more complicated equilibria. The circular cylinder of the next section is a typical example.

8. Cylinder. In this section the body is a circular cylinder of radius R and height H.

In one equilibrium position the cylinder floats on end, with $\theta = 0$ or $\theta = \pi$ in the notation of Section 7. A simple calculation of the pitch metacenter shows that this orientation is stable if

$$\rho(1-\rho)\left(\frac{H}{R}\right)^2 < \frac{1}{2}. \tag{9}$$

In another equilibrium the cylinder floats on its side, with $\theta = \pi/2$. The waterline figure A is a rectangle, with sides H and w, where Archimedes' principle determines w as a root of

$$\arcsin\frac{w}{2R} - \frac{w}{2R}\sqrt{1-\left(\frac{w}{2R}\right)^2} = \rho\pi.$$

The corollary to Theorem 3 shows that this equilibrium is stable if

$$w < H. \tag{10}$$

Both (9) and (10) may hold if ρ is small, only (9) if H/R is small, only (10) if H/R is large, and for some parameter values (e.g., $\rho = 1/2$, $H/R = 1.7$) neither (9) nor (10) hold. Then some cylinders must float with axis tilted at other angles θ, not 0, $\pi/2$, or π. There are several cases to consider because the waterline may intersect either end of the cylinder, neither, or both. In what follows x, y, z are cartesian coordinates, fixed in the body, with the z-axis placed along the axis of revolution. Then the cylinder is $x^2 + y^2 \leqslant R^2$, $0 \leqslant z \leqslant H$.

In the simplest configuration, the end $z = 0$ is completely submerged and the end $z = H$ is completely out of the water. The waterline plane passes through the points $(-R, 0, \rho H - R\tan\theta)$, $(0, 0, \rho H)$, and $(R, 0, \rho H + R\tan\theta)$. This configuration is possible only if

$$\frac{R}{H}\tan\theta \leqslant \operatorname{Min}\{\rho, 1-\rho\}. \tag{11}$$

If this orientation is an equilibrium, it is stable. For, the waterline figure A is an ellipse with semiaxes R and $R\sec\theta > R$; the corollary to Theorem 3 ensures pitch-stability. The equilibrium condition is that G be directly above B, i.e., the line GB must make angle θ with the axis of the cylinder. Now $G = (0, 0, H/2)$ and $B = (B_x, 0, B_z)$ where

$$B_x = \frac{R^2\tan\theta}{4\rho H}, \quad B_z = \frac{\rho H}{2}\left\{1 + \left(\frac{R\tan\theta}{2\rho H}\right)^2\right\}.$$

The condition for stability is $B_x = (H/2 - B_z)\tan\theta$, or

$$\rho(1-\rho)\left(\frac{H}{R}\right)^2 = \frac{1}{2} + \frac{1}{4}\tan^2\theta. \tag{12}$$

If $\rho(1-\rho)(H/R)^2 < 1/2$, there is no equilibrium of this kind, but then (9) shows that the cylinder floats stably on its end. For larger values, (12) determines an angle θ. If θ satisfies (11), then the cylinder floats stably tilted at angle θ. If (11) is violated, (12) no longer applies and the equilibrium is one with waterline passing through one or both ends.

To simplify the problem, only the case $\rho = 1/2$ will be considered. Formula (12) with $\rho = 1/2$ determines an angle θ satisfying (11) if $2 < (H/R)^2 < 8/3$. When $H/R = \sqrt{8/3} = 1.63299$, the tilt angle is $\theta = 39.23°$ and the two ends of the cylinder are just tangent to the water surface. For larger values of H/R, consider a configuration with waterline plane passing through $(-(H/2)\cot\theta, 0, 0)$, $(0, 0, H/2)$, and $((H/2)\cot\theta, 0, H)$. The formulas that follow are simplified by the notation

$$t = \frac{H}{2R}\cot\theta, \quad J_k = \int_0^t x^k\sqrt{1-x^2}\,dx \tag{13}$$

for $k = 0, 2$. The integrals J_k are elementary,

$$J_0 = \frac{1}{2}\{t\sqrt{1-t^2} + \arcsin t\}$$

$$J_2 = \frac{1}{4}\{J_0 - t(1-t^2)^{3/2}\}.$$

The waterline in this configuration passes through both faces. That requires $(H/2)\cot\theta < R$, i.e., $t < 1$.

Now $V = \pi R^2 H/2$ and $B = (B_x, 0, B_z)$ with

$$VB_x = \frac{4}{3}R^4(J_0 - J_2)$$

$$VB_z = R^2H^2\left(\frac{\pi}{4} - \frac{1}{2}J_0\right) + 2R^4\tan^2\theta J_2.$$

The equilibrium condition

$$B_x = \left(\frac{H}{2} - B_z\right)\tan\theta$$

is

$$\tan^2\theta = \frac{2}{3}\frac{J_0 - J_2}{J_0t^2 - J_2}. \tag{14}$$

The stability of this equilibrium must be determined by finding the principle moments of inertia of the waterline figure A. A is no longer the entire ellipse, with semiaxes R and $R\sec\theta$, because that ellipse protrudes beyond the ends of the cylinder. To get A, pieces at the ends of the ellipse must be cut off to leave a figure of length $H\csc\theta$ (see FIGURE 5). The moments for pitching and rolling are

$$I_p = 4R^4\sec^3\theta J_2$$

$$I_r = \tfrac{4}{3}R^4\sec\theta(J_0 - J_2).$$

The corollary to Theorem 3 now gives the stability condition $I_p > I_r$, or

$$\cos^2\theta < \frac{3J_2}{J_0 - J_2}. \tag{15}$$

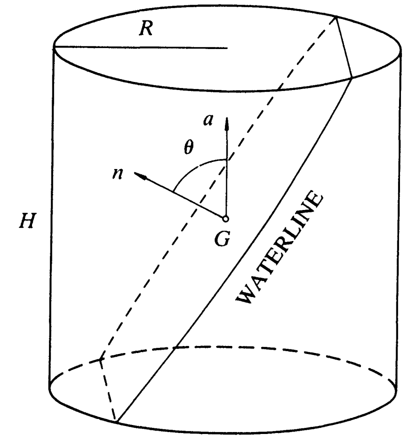

FIG. 5. Waterline figure A of floating circular cylinder.

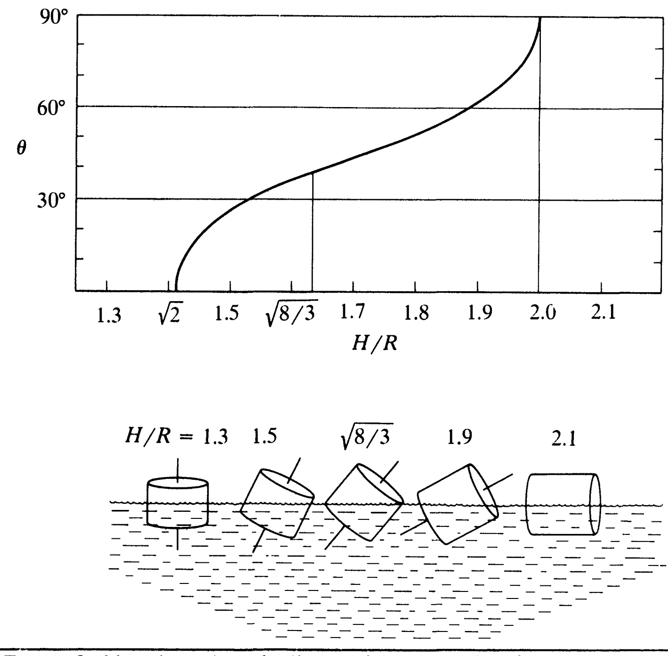

FIG. 6. Stable orientation of cylinder of density 1/2 as function of H/R.

Each value of t in $0 < t < 1$ determines an angle θ in (14) and then (13) gives $H/R = 2t \tan \theta$. A check on the condition (15) shows that the equilibrium is stable throughout $0 < t < 1$. As t varies, H/R and θ range over $2 > H/R > \sqrt{8/3}$ and $90° > \theta > 39.23°$.

The cylinder with density $\rho = 1/2$ has, for each H/R, a unique angle θ at which it floats stably. FIGURE 6 shows how the stable orientation varies with H/R. Formulas $\theta = 0°$, (12), (14), and $\theta = 90°$ hold in the ranges $0 < H/R < \sqrt{2}$, $\sqrt{2} < H/R < \sqrt{8/3}$, $\sqrt{8/3} < H/R < 2$ and $2 < H/R < \infty$.

9. Inversion symmetry. A body has *inversion symmetry* if, when the origin O of coordinates is placed at the centroid G, the inversion $(x, y, z) \to (-x, -y, -z)$ leaves the body unchanged. If a body has inversion symmetry, then any plane through G cuts the body into two congruent halves. If a body has $\rho = 1/2$, as well as inversion symmetry, then its waterlines are easy to find. The waterline, for any given n, lies in the plane normal to n that passes through G.

Bodies of revolution, being already indifferently stable with respect to rolling, seem likely candidates to be indifferently stable to all perturbations, as requested by Ulam.

THEOREM 4. *Except for spheres, there are no indifferently stable bodies of revolution with* $\rho = 1/2$ *that also have inversion symmetry.*

Proof. Introduce spherical polar coordinates (r, γ, δ) with origin at G and with polar angle γ (the axis of revolution is $\gamma = 0$ and $\gamma = \pi$). An equation of form $r = r(\gamma)$ will describe the surface of the body.

Consider the body floating with upward direction vector n and let n have angle coordinates $\gamma = \theta$, $\delta = 0$. Now introduce a second system of spherical polar coordinates (r, α, β) with pole $\alpha = 0$ at n. The original polar angle γ is related to the new coordinates by

$$\cos\gamma = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos\beta. \tag{16}$$

The potential $U(n)$ is (1) with

$$S(n) = -\iiint (P \cdot n) r^2 \sin\alpha \, dr \, d\alpha \, d\beta,$$

wherein the integration extends over points P of the body below the waterline plane. The range of integration is

$$0 \leqslant r \leqslant r(\gamma), \quad 0 \leqslant \beta \leqslant 2\pi, \quad \frac{\pi}{2} \leqslant \alpha \leqslant \pi,$$

where γ depends on α and β, as in (16). Because $P \cdot n = r\cos\alpha$, the r-integration is immediate;

$$S(n) = -\frac{1}{4}\iint r^4(\gamma)\cos\alpha\sin\alpha \, d\alpha \, d\beta. \tag{17}$$

Expand $r^4(\gamma)$ in a Neumann series

$$r^4(\gamma) = \sum A_k P_k(\cos\gamma) \tag{18}$$

with $P_k(\cdot)$ the kth Legendre polynomial. Because the body has inversion symmetry, $r^4(\pi - \gamma) = r^4(\gamma)$, and coefficients A_k with odd k vanish. With $\cos\gamma$ given by (16) the addition theorem for Legendre polynomials ([**3**]) expresses $P_k(\cos\gamma)$ in terms of Legendre polynomials $P_k(\cdot)$ and associated Legendre polynomials $P_k^m(\cdot)$ of $\cos\alpha$ and $\cos\theta$

$$P_k(\cos\gamma) = P_k(\cos\alpha)P_k(\cos\theta) + 2\sum_{m=1}^{k}\frac{(k-m)!}{(k+m)!}P_k^m(\cos\theta)P_k^m(\cos\alpha)\cos m\beta.$$

Then when (18) is substituted into (17), the integration on β removes the terms containing associated Legendre polynomials, and leaves

$$S(n) = -\frac{\pi}{2}\left\{A_0 + \sum A_k P_k(\cos\theta)\int_0^1 P_k(x)\,x\,dx\right\}. \tag{19}$$

Now suppose the body is indifferently stable, so that $S(n)$ is constant. In (19), the coefficient of each term $P_k(\cos\theta)$ must vanish. With k even, say $k = 2r \geqslant 2$,

$$\int_0^1 P_k(x)\,x\,dx = \frac{(-1)^{r+1}}{4^r(2r-1)(2r+1)}\binom{2r}{r} \neq 0$$

and so $A_k = 0$. Also, as was mentioned above, $A_k = 0$ for odd k because of the inversion symmetry. Then (18) becomes $r^4(\gamma) = A_0$ and the body is a sphere. ■

Bodies with inversion symmetry are so special that the theorem does not strongly suggest that the sphere may be the only indifferently stable body. For example, in the problem of indifferently roll-stable cylinders, a very similar argument would show there are no non-circular examples with inversion symmetry. Yet, Auerbach produced examples without inversion symmetry.

10. Cube. This section will show that a homogeneous cube can float in surprising ways. Again, x, y, z will be body coordinates, now in the interval $[0, 1]$. Then $G = (1/2, 1/2, 1/2)$. Three planes through G, each normal to a coordinate axis, are planes of mirror symmetry of the cube. It follows, with $n = (\lambda, \mu, \nu)$, that $U(n)$ depends only on the magnitudes $|\lambda|, |\mu|, |\nu|$. In what follows λ, μ, ν will be assumed non-negative. It will also suffice to restrict ρ to $0 \leqslant \rho \leqslant 1/2$ because Theorem 2 will then show how denser cubes behave. Equilibria with n along axes of symmetry will be considered first; the metacenters for these equilibria are easily calculated.

When the cube floats face-down, $n \cdot (G - B) = (1 - \rho)/2$. Also $n \cdot (M - B) = 1/(12\rho)$ for rotations about any axis t. Then the cube floats stably on its face if $\rho(1 - \rho) < 1/6$, i.e., if $\rho < .211325$.

When the cube floats edge-down,

$$n \cdot (G - B) = \frac{1}{\sqrt{2}} - \frac{2}{3}\sqrt{\rho}\,.$$

The metacenter now depends on the axis t. It will be convenient to use the terms roll and pitch as though the cube were a ship with the submerged edge as its keel. The metacenters M_r, M_p for rolling and pitching satisfy

$$n \cdot (M_r - B) = \frac{2}{3}\sqrt{\rho}\,, \quad n \cdot (M_p - B) = \frac{1}{6\sqrt{\rho}}\,.$$

For $0 \leqslant \rho < 1/8$ the equilibrium is a saddle point of $U(n)$, roll-unstable and pitch-stable. For $1/8 < \rho < 9/32$ it is a maximum. For $9/32 < \rho \leqslant 1/2$ it is again a saddle point, but roll-stable and pitch-unstable.

When the cube floats vertex-down the submerged part of the body is a triangular pyramid for $0 < \rho \leqslant 1/6$. But, when $1/6 < \rho < 1/2$ the waterline plane cuts all 6 faces of the cube and the figure A is an irregular hexagon. The calculations of B and M are longer, although still elementary. Floating vertex-down turns out to be unstable (a maximum) for $0 \leqslant \rho < 1/6$ and stable for $1/6 < \rho \leqslant 1/2$.

Equation (5) shows that axes of symmetry need not be the only equilibrium directions. When $1/8 < \rho < 9/32$ these axes provide no saddle points. A curious phenomenon occurs when $.211325 < \rho < .224602$. In that range the potential $U(n)$ of the unstable face-down orientation is actually lower than the potential of the stable vertex-down orientation; then there must be other stable orientations with even lower potential.

To find the missing equilibria numerically one can just plot $U(n)$ as a function of n on the unit sphere. The function $U(n)$, which will be derived next, can only be piecewise analytic; its form depends on which vertices are submerged.

For fixed $n = (\lambda, \mu, \nu)$, the derivation introduces an integral

$$F(s) = \iiint e^{-(\lambda x + \mu y + \nu z)s}\, dx\, dy\, dz$$

taken over the entire body. For the unit cube,

$$F(s) = \frac{1 - e^{-\lambda s}}{\lambda s} \frac{1 - e^{-\mu s}}{\mu s} \frac{1 - e^{-\nu s}}{\nu s}, \tag{20}$$

but the same argument will apply to a general body as long as the origin is located so that the body lies entirely in the positive octant.

THEOREM 5. *For $k = 1, 2$, write $f_k(h)$ for the inverse Laplace transform of $F(s)/s^k$. When the water surface is the plane $\lambda x + \mu y + \nu z = h$, $f_1(h)$ is the displaced volume V and $f_2(h)/V$ is the depth of the center of buoyancy B below the water level.*

Proof. Write $A(h)$ for the area of intersection of the plane $\lambda x + \mu y + \nu z = h$ with the body. Then

$$F(s) = \iiint e^{-hs}\, dx\, dy\, dz = \int_0^\infty e^{-hs} A(h)\, dh,$$

the Laplace transform of $A(h)$. $F(s)/s$ is the inverse Laplace transform of

$$f_1(h) = \int_0^h A(h')\, dh',$$

but that integral is just the displaced volume V below the plane $\lambda x + \mu y + \nu z = h$.

Likewise, $F(s)/s^2$ is the Laplace transform of

$$f_2(h) = \int_0^h \int_0^{h'} A(h'')\, dh''\, dh'$$
$$= \int_0^h (h - h'') A(h'')\, dh'',$$

the second line following by interchanging the order of integration. But this integral is just V times the depth of the center of buoyancy. ∎

In general, Archimedes' principle specifies V. Then the equation

$$f_1(h) = V, \tag{21}$$

from Theorem 5, determines h and the depth $f_2(h)/V$ of B. G is at height $n \cdot G$ above the origin, or $n \cdot G - h$ above the water level. Then the potential $U(n)$ in (1) is proportional to

$$n \cdot (G - B) = n \cdot G + \frac{f_2(h)}{V} - h. \tag{22}$$

For the unit cube, $V = \rho$ in (21) and $n \cdot G = (\lambda + \mu + \nu)/2$ in (22). To find $f_k(h)$, multiply out (20) into a sum of 8 terms, each an exponential divided by $\lambda\mu\nu s^{k+3}$. Transforming this expression term-by-term produces

$$f_k(h) = \frac{1}{\lambda\mu\nu(k+2)!}\{\|h\|^{k+2} - \|h - \lambda\|^{k+2} - \|h - \mu\|^{k+2} - \|h - \nu\|^{k+2}$$
$$+\|h - \lambda - \mu\|^{k+2} + \|h - \mu - \nu\|^{k+2} + \|h - \nu - \lambda\|^{k+2}$$
$$-\|h - \lambda - \mu - \nu\|^{k+2}\}, \tag{23}$$

where $\|z\|$ means z if $z \geqslant 0$ and 0 if $z < 0$. Each term in (23) represents a different vertex of the cube. Thus h is the depth of $(0, 0, 0)$, $h - \lambda$ is the depth of $(1, 0, 0)$, $h - \lambda - \mu$ is the depth of $(1, 1, 0)$, etc. Only the submerged vertices contribute non-zero terms to (23).

Formulas (21), (22), (23) were used in a numerical study of a cube of density $\rho = .217$. This density lies in the interesting range, discussed earlier, in which the global minimum of $U(n)$ occurs away from the cube's symmetry axes and so remains to be found. FIGURE 7 shows level contours of $n \cdot (G - B)$, drawn in gnomonic projection from G onto a typical face of the cube as in FIGURE 4b. Beside the equilibria at the face center, edge centers, and vertices, there are four more saddle points, one near each corner of the face. Note that equipotentials in Figure 7 do not meet the face edges at right angles because the gnomonic projection is not conformal.

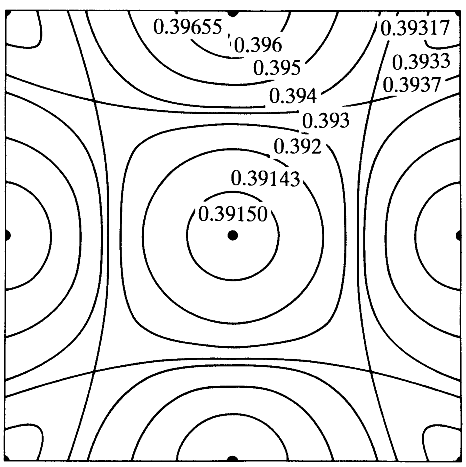

FIG. 7. Equipotential lines on a face of a cube of density $\rho = .217$.

The missing global minima turns out not to be isolated points. They form a circle, with $n \cdot (G - B) = .39143$, surrounding the face center. To show this analytically, consider directions n near $(0, 0, 1)$, a face-center axis. Suppose λ and μ small enough (and $\nu = \sqrt{1 - \lambda^2 - \mu^2}$ large enough) so that the submerged vertices are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(1,1,0)$. Then $f_k(h)$ in (23) is a sum of four terms. When the terms of (23) are expanded in powers of h, considerable cancellation occurs, leaving

$$f_1(h) = \left(h - \frac{\lambda + \mu}{2}\right) \Big/ \nu$$

and

$$f_2(h) = \frac{1}{12\nu}\{6h^2 - 6h(\lambda + \mu) + 2\lambda^2 + 3\lambda\mu + 2\mu^2\}. \tag{24}$$

The condition $f_1(h) = V = \rho$ of Theorem 5 provides

$$h = \frac{\lambda + \mu}{2} + \rho\nu. \tag{25}$$

Now substitute (24) into (22) and use (25) to eliminate h. The result, with the help of $\lambda^2 + \mu^2 + \nu^2 = 1$, simplifies to

$$n \cdot (G - B) = \frac{1}{24\rho}\left\{\frac{1}{\nu} + \nu[12\rho(1 - \rho) - 1]\right\} \tag{26}$$

Only the ν component of n enters into (26) and so the level contours of $n \cdot (G - B)$ near the face center are circles. If $\rho(1 - \rho) > 1/6$ (recall that this was also the condition that the face center be an unstable equilibrium point), then $n \cdot (G - B)$ in (26) is a minimum at

$$\nu = \frac{1}{\sqrt{12\rho(1 - \rho) - 1}}$$

With $\rho = .217$ the minimum occurs at $\nu = .981085 = \cos 11.16°$. For all such n, only the vertices of one face are submerged and (24) does apply.

A cube of density $\rho = .217$ will float with a face-center tilted 11.16° away from vertical. However, the coordinates λ and μ of n are restricted only by $\lambda^2 + \mu^2 = 1 - \nu^2 = .037473$ and the slightest disturbance will move the cube to a new equally stable orientation with the same ν. This cube resembles the cylinder in having a one-parameter family of indifferently stable orientations. In fact, as n ranges over all directions, $n \cdot (G - B)$ remains within 0.65% of .3940 and so the cube comes close to satisfying the condition $n \cdot (G - B) = \text{const}$ of an indifferently stable body.

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