

§5.1-5.3 Test Solutions
5/10/16

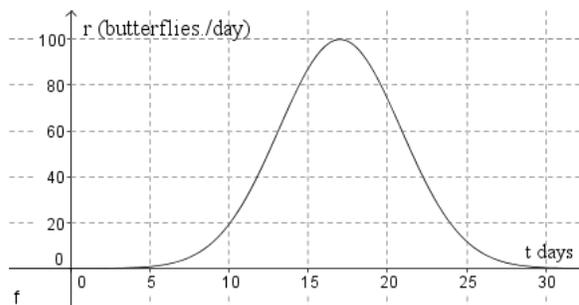
1. (16 points) The graph below shows the rate of butterfly births in a Monarch butterfly nest over a period of thirty days.

- (a) Approximate the area under the curve using a partition of $[0, 30]$ with 3 subintervals of equal length and midpoints as sample points. Approximate the function values from the graph.

Solution: $A \approx 10 \cdot (1 + 85 + 10) = 960$

- (b) Explain what the integral $\int_0^{30} f(t) dt$ means in terms of the function $r = f(t)$.

Solution: The integral computes the total number of butterflies born over the 30 day period.



2. (21 points) The speed, v , of a runner is measured at various times, t , to produce the tabulated values:

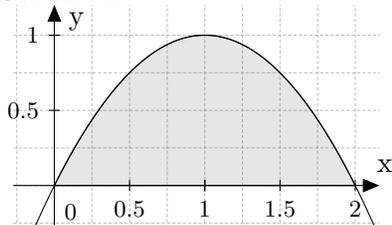
t (sec)	0	1	2	3	4	5	6
v (m/s)	1	3	5	6	7	8	8

- (a) Approximate the distance the runner has traveled in these six seconds using three subintervals of equal length and right endpoints as sample points. **Solution:** $2 \cdot (5 + 7 + 8) = 40\text{m}$.
- (b) Approximate the distance the runner has traveled in these six seconds using three subintervals of equal length and left endpoints as sample points. **Solution:** $2 \cdot (1 + 5 + 7) = 26\text{m}$.
- (c) Approximate the distance the runner has traveled in these six seconds using three subintervals of equal length and midpoints as sample points. **Solution:** $2 \cdot (3 + 6 + 8) = 34\text{m}$.

3. (28 points) Consider the area bounded by $f(x) = 1 - (x - 1)^2$ and the x -axis.

- (a) Draw a diagram illustrating this region.

Solution:



- (b) Approximate the area using a partition with 3 intervals of equal length and midpoints as sample points.

Solution: $\Delta x = \frac{2-0}{3} = \frac{2}{3}$ so $x_i = 0 + i \cdot \frac{2}{3} = \frac{2i}{3}$ with $x_i^* = \frac{2i-1}{3}$ and the approximating sum is $A \approx \frac{2}{3} \left(1 - \left(\frac{1}{3} - 1\right)^2 + 1 - (1 - 1)^2 + 1 - \left(\frac{5}{3} - 1\right)^2 \right) = \frac{2}{3} \left(3 - 2 \cdot \frac{4}{9} \right) = \frac{38}{27} = 1.\overline{407}$

- (c) Use the definition of the definite integral to compute the area as the limit of a Riemann sum. Do *not* use the Fundamental Theorem of Calculus.

Solution: Note that $f(x) = 2x - x^2$ is a simpler form of the function for this computation.

Using $\Delta x = \frac{2-0}{n} = \frac{2}{n}$ so $x_i = 0 + i \cdot \frac{2}{n} = \frac{2i}{n}$, $A = \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \left(\frac{4i}{n} - \frac{4i^2}{n^2} \right) = \lim_{n \rightarrow \infty} \frac{8}{n^2} \left(\sum_{i=1}^n i - \frac{1}{n} \sum_{i=1}^n i^2 \right) = \lim_{n \rightarrow \infty} \frac{8}{n^2} \left(\frac{n^2+n}{2} - \frac{1}{n} \frac{2n^3+3n^2+n}{6} \right)$. Using "little o" notation, $A = \lim_{n \rightarrow \infty} \left(4 - \frac{8}{3} + o\left(\frac{1}{n}\right) \right) = \boxed{\frac{4}{3}}$.

- (d) Is the midpoints approximation an underestimate or an overestimate? Why does that make sense? ANS: The approximating sum in (b) is a slight overestimate, which makes sense since the curve is concave down.

4. (35 points) Evaluate:

$$(a) \int_0^{\pi} \frac{d}{dx} \sin x^2 dx$$

Solution: Let $u = \sin x^2$ then $\frac{du}{dx} \cdot dx = du$. Then $x = 0 \Rightarrow u = 0$ and $x = \pi \Rightarrow u = \sin \pi^2$ so that

$$\int_0^{\pi} \frac{d}{dx} \sin x^2 dx = \int_0^{\sin \pi^2} du = u \Big|_0^{\sin \pi^2} = \sin \pi^2$$

$$(b) \frac{d}{dx} \int_0^{x^2} \sin t^2 dt$$

Solution: Let $u = x^2$ so

$$\frac{d}{dx} \int_0^{x^2} \sin t^2 dt = \frac{du}{dx} \frac{d}{du} \int_0^u \sin t^2 dt = \frac{du}{dx} \cdot \sin u^2 = 2x \sin x^4$$

$$(c) \frac{d}{dx} \int_0^{\pi} \sin t^2 dt$$

Solution: Since the definite integral $\int_0^{\pi} \sin t^2 dt$ is a constant, $\frac{d}{dx} \int_0^{\pi} \sin t^2 dt = 0$

$$(d) \int_1^{\sqrt{3}} \frac{2}{1+x^2} dx = 2 \arctan(x) \Big|_1^{\sqrt{3}} = 2 \arctan(\sqrt{3}) - \arctan(1) = 2 \frac{\pi}{3} - 2 \frac{\pi}{4} = \frac{\pi}{6}$$

$$(e) \frac{d}{dx} \int_{\sin x}^{\cos x} \frac{1-t^2}{1+t^2} dt = \frac{d}{dx} \int_0^{\cos x} \frac{1-t^2}{1+t^2} dt - \int_0^{\sin x} \frac{1-t^2}{1+t^2} dt = -\sin(x) \frac{\sin^2 x}{1+\cos^2 x} - \cos x \frac{\cos^2 x}{1+\sin^2 x}$$